Understanding and using SAT and SMT solvers

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14th Summer School on Formal Techniques
May 24-30, 2025

The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

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(p_1 \geq 5 \lor p_2 \geq 5) \land p_3 \geq 10 \land p_1 + 2p_2 + 5p_3 \leq 180 \land \\
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Logic:

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Logic: Linear integer arithmetic.

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Eve is eager to make scientific visits.

- She has 100 travel wishes $A_1, \ldots, A_{100}$.
- She is allowed to make only 5 travels.
- She wants to be physically at A_1.
- To coordinate a project, she needs to visit either A_2 or A_3.
- Travel A_i costs C_i EUR.
- Eve can spend up to C EUR.
- Travel A_i takes T_i days.
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$$\left(\bigwedge_{i=1}^{100} \Big((a_i = 0 \wedge c_i = 0 \wedge t_i = 0) \vee (a_i = 1 \wedge c_i = C_i \wedge t_i = T_i) \Big) \right) \wedge$$

$$\left(\sum_{i=1}^{100} a_i \leq 5 \right) \wedge (a_1 = 1) \wedge (a_2 = 1 \vee a_3 = 1) \wedge \left(\sum_{i=1}^{100} c_i \leq C \right) \wedge \left(\sum_{i=1}^{100} t_i \geq T \right)$$

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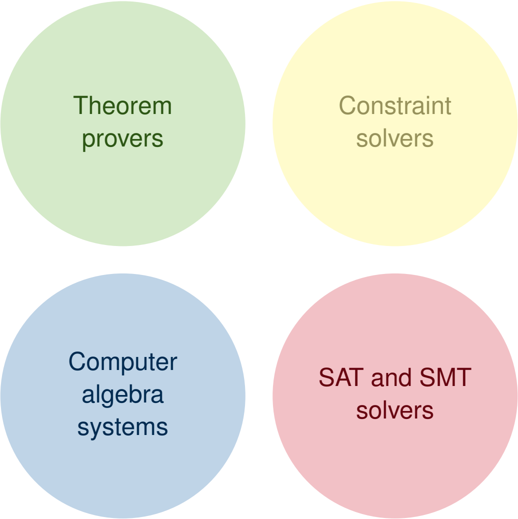
Logic: Mixed integer linear arithmetic.

Some technologies for satisfiability checking



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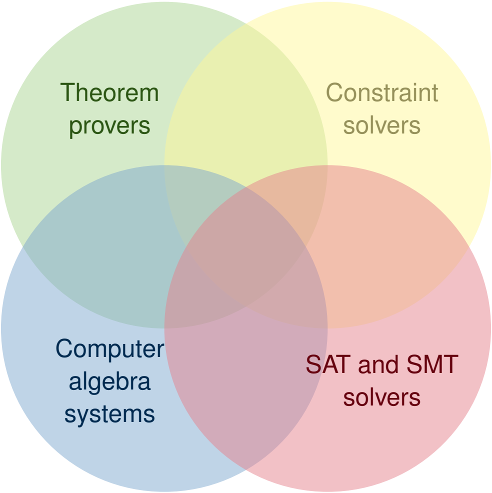

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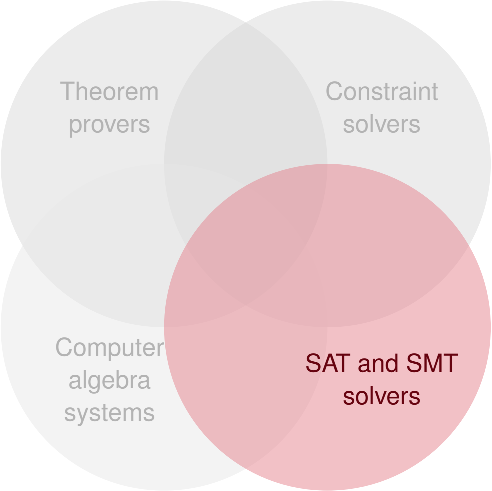
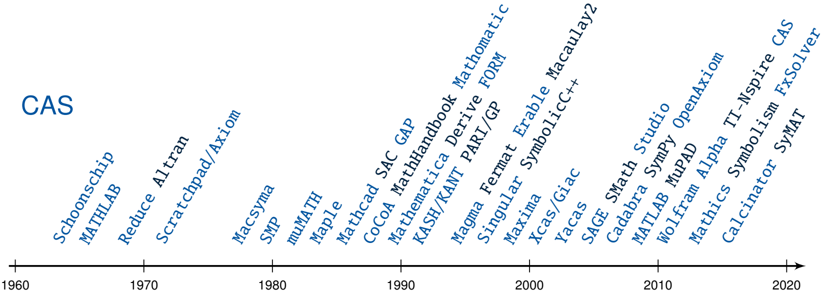


Tool development

CAS



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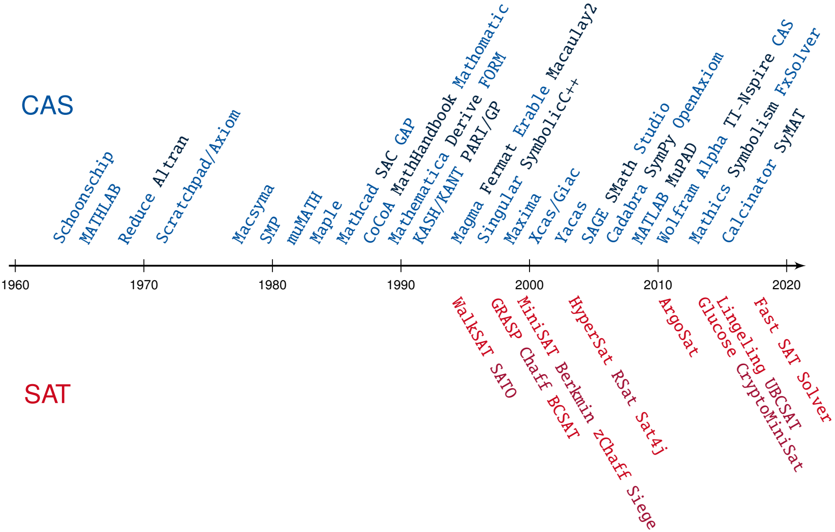

Satisfiability checking for propositional logic

Success story: SAT-solving

- Practical problems with millions of variables are solvable.
- A wide range of applications, e.g., verification, synthesis, combinatorial optimisation, etc.

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Community support:

- Standard input language.
- Large benchmark library.
- Competitions since 2002.
- SAT Live! forum as community platform, dedicated conferences, journals, etc.

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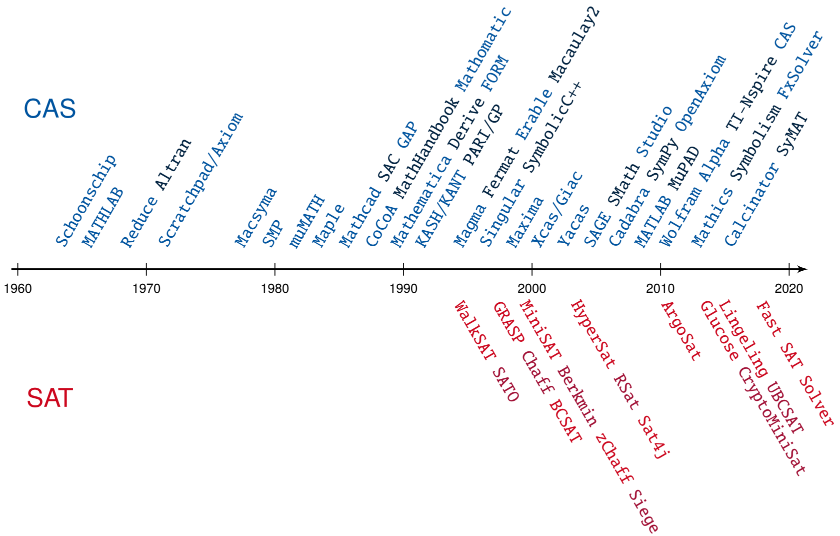

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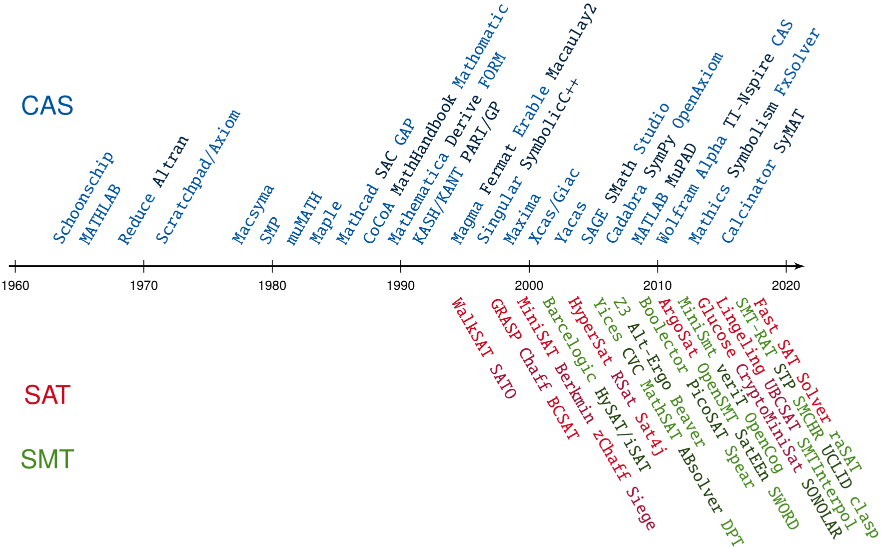

Satisfiability modulo theories (SMT) solving:

- Propositional logic is sometimes too weak for modelling.
- Increase expressiveness: quantifier-free (QF) fragments of first-order logic over various theories.

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Community support:

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Contents

Syntax of propositional logic

Abstract syntax of well-formed propositional logic formulae:

$$\varphi \; := \; a \; | \; (\neg\varphi) \; | \; (\varphi \wedge \varphi)$$

where *AP* is a set of (atomic) propositions (Boolean variables) and $a \in AP$.

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Semantics of propositional logic

- Truth tables define the semantics (=meaning) of the operators.
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p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \oplus q$
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
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Each possible assignment is covered by a line of the truth table.
α satisfies φ iff in the line for α and the column for φ the entry is 1.

Conjunctive normal form

- A literal is either a variable or the negation of a variable.
- A clause is a disjunction of literals.
- A formula in Conjunctive Normal Form (CNF) is a conjunction of clauses.

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- A clause is a disjunction of literals.
- A formula in Conjunctive Normal Form (CNF) is a conjunction of clauses.
- Every propositional logic formula can be converted to an equi-satisfiable CNF in linear time and space on the cost of (linearly many) new variables.

Tseitin's CNF encoding

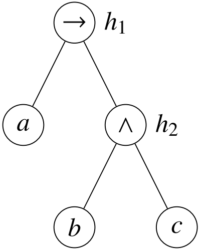
Consider the formula $\varphi = (a \rightarrow (b \wedge c))$.

Tseitin's encoding:

$(h_1 \leftrightarrow (a \rightarrow h_2))\wedge$

$(h_2 \leftrightarrow (b \wedge c))\wedge$

(h_1)



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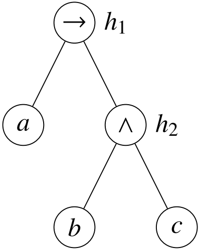

- Each node's encoding has a CNF representation with 3 or 4 clauses.

  $h_1 \leftrightarrow (a \rightarrow h_2)$ in CNF: $(h_1 \vee a) \wedge (h_1 \vee \neg h_2) \wedge (\neg h_1 \vee \neg a \vee h_2)$
  $h_2 \leftrightarrow (b \wedge c)$ in CNF: $(\neg h_2 \vee b) \wedge (\neg h_2 \vee c) \wedge (h_2 \vee \neg b \vee \neg c)$

Contents

Satisfiability problem

Given:

- Propositional logic formula φ in CNF.

Question:

- Is φ satisfiable?

  (Is there a model for φ?)

1. Exploration: Exploration

$(a \vee c) \wedge (\neg a \vee c) \wedge (a \vee \neg c) \wedge (\neg a \vee \neg c) \wedge (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b \vee \neg c)$

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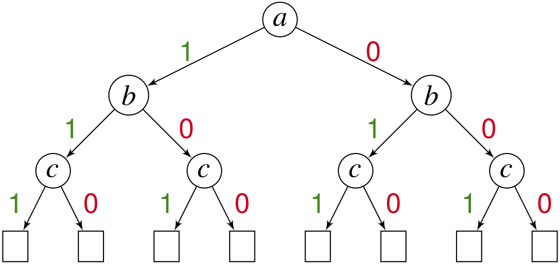

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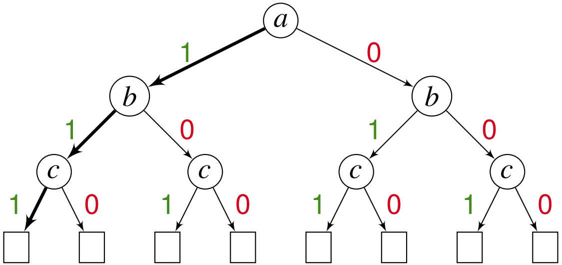

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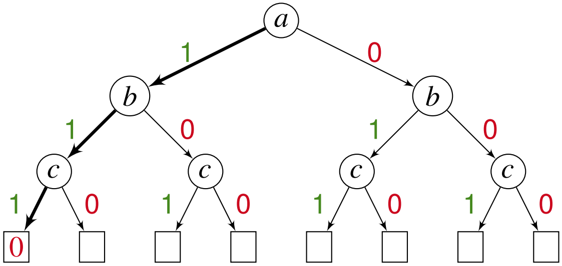

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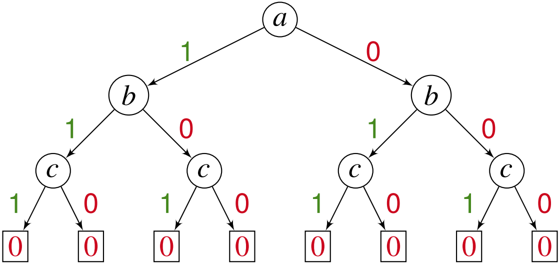
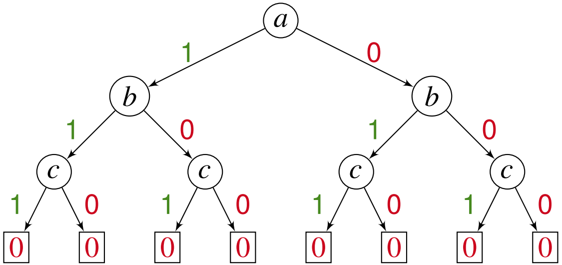
$(a \lor c) \land (\neg a \lor c) \land (a \lor \neg c) \land (\neg a \lor \neg c) \land (a \lor b) \land (\neg a \lor b) \land (a \lor \neg b \lor \neg c)$



$$(a \vee c) \wedge (\neg a \vee c) \wedge (a \vee \neg c) \wedge (\neg a \vee \neg c) \wedge (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b \vee \neg c)$$



unsatisfiable problem in n variables $\rightsquigarrow$ ALWAYS 2^n assignments need to be tested

2. Proof system: Boolean resolution

$$(a \vee c) \wedge (\neg a \vee c) \wedge (a \vee \neg c) \wedge (\neg a \vee \neg c) \wedge (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b \vee \neg c)$$

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$$\frac{(x \vee \ell_1 \vee \ldots \vee \ell_n) \qquad (\neg x \vee \ell'_1 \vee \ldots \vee \ell'_m)}{(\ell_1 \vee \ldots \vee \ell_n \vee \ell'_1 \vee \ldots \vee \ell'_m)} \; \texttt{Rule}_{\texttt{res}}$$

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$(b \vee c)$

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(true)

$$\frac{(\, x \vee \ell_1 \vee \ldots \vee \ell_n \,) \qquad (\, \neg x \vee \ell'_1 \vee \ldots \vee \ell'_m \,)}{(\, \ell_1 \vee \ldots \vee \ell_n \;\vee\; \ell'_1 \vee \ldots \vee \ell'_m \,)} \; \mathtt{Rule}_{\mathtt{res}}$$

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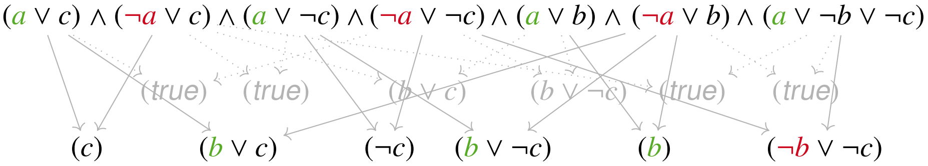
$$\exists x. \quad C \wedge C_x \wedge C_{\neg x}$$

$$\equiv$$

$$C \wedge \bigwedge_{c_x \in C_x} \bigwedge_{c_{\neg x} \in C_{\neg x}} \textit{resolvent}(c_x, c_{\neg x}, x)$$



$$\frac{(\,x \lor \ell_1 \lor \ldots \lor \ell_n\,) \qquad (\,\neg x \lor \ell'_1 \lor \ldots \lor \ell'_m\,)}{(\,\ell_1 \lor \ldots \lor \ell_n \;\lor\; \ell'_1 \lor \ldots \lor \ell'_m\,)}\ \texttt{Rule}_{\texttt{res}}$$

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$$C \;\land\; \bigwedge_{c_x \in C_x} \;\bigwedge_{c_{\neg x} \in C_{\neg x}} \textit{resolvent}(c_x, c_{\neg x}, x)$$

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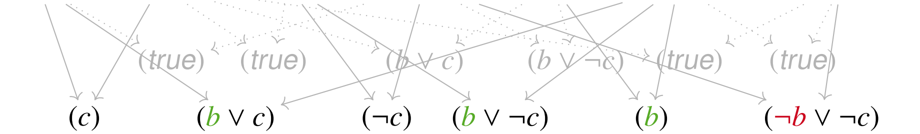

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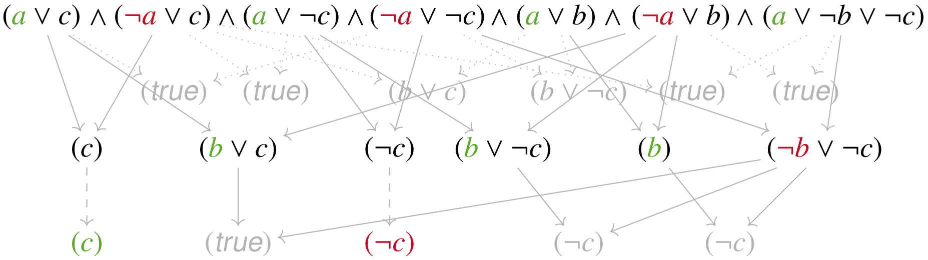
2. Proof system: Boolean resolution



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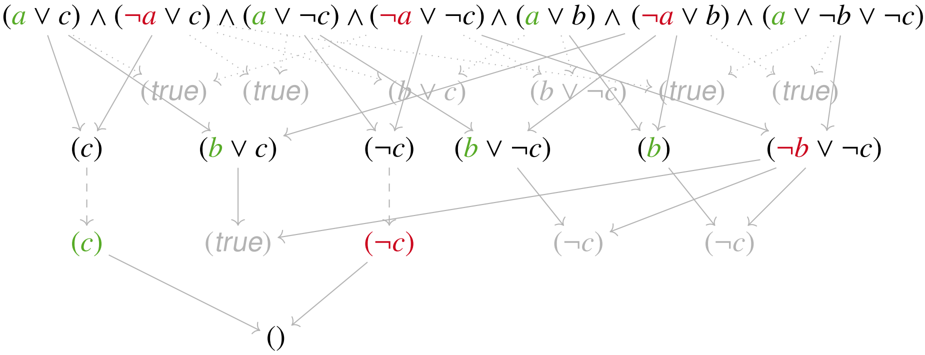

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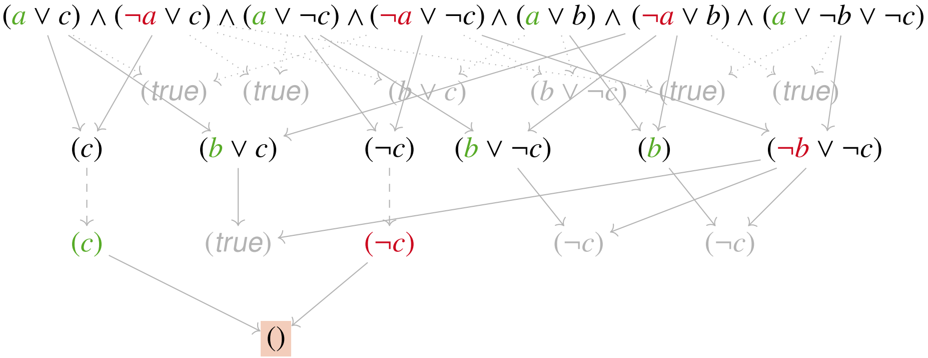

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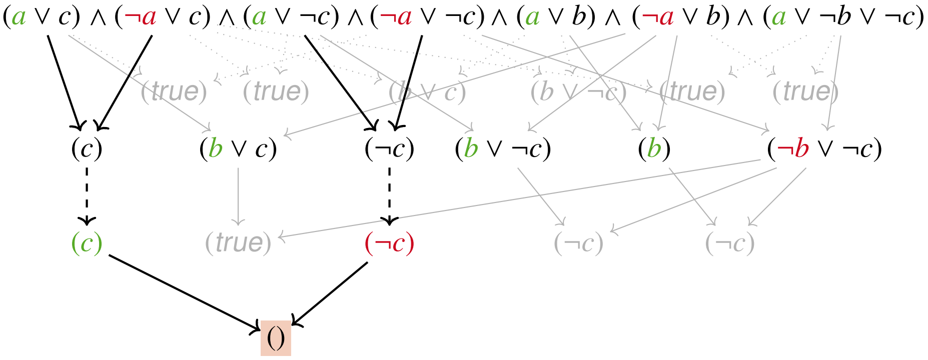

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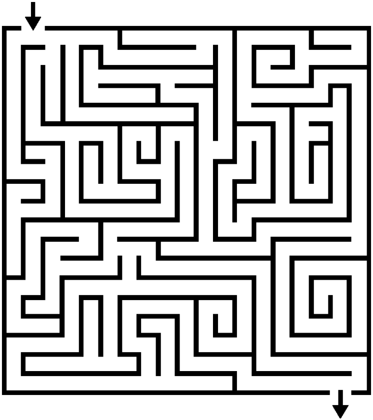
SAT solving: The DPLL+CDCL idea

[Davis et al., '60/61] [Marques-Silva et al., '96]



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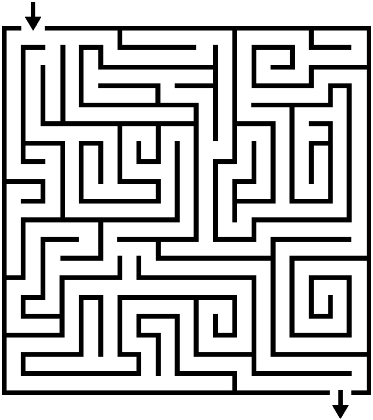

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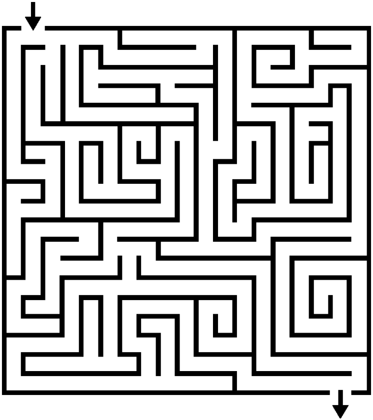

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Exploration

SAT solving: The DPLL+CDCL idea

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Exploration

Look-ahead

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Exploration

Look-ahead

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The DPLL+CDCL algorithm

```
if (!BCP()) return UNSAT;
while (true)
{
        if (!decide()) return SAT;
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- Assume in the following: all literals in a clause have different variables

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satisfied:	at least one literal is satisfied
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unit:	all but one literals are assigned but none are satisfied
unresolved:	all other cases

Example :

x_1	x_2	x_3	$c = (x_1 \vee x_2 \vee x_3)$
1	0		satisfied
0	0	0	unsatisfied
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BCP: Unit clauses are used to imply consequences of decisions.

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Some notations:

Decision Level (DL) is a counter for decisions

Antecedent(ℓ): unit clause implying the value of literal ℓ (nil if decision)

DPLL: Exploration with propagation

$$\underbrace{(\neg x \vee y \vee z)}_{c_1} \wedge \underbrace{(y \vee \neg z)}_{c_2} \wedge \underbrace{(\neg x \vee \neg y)}_{c_3}$$

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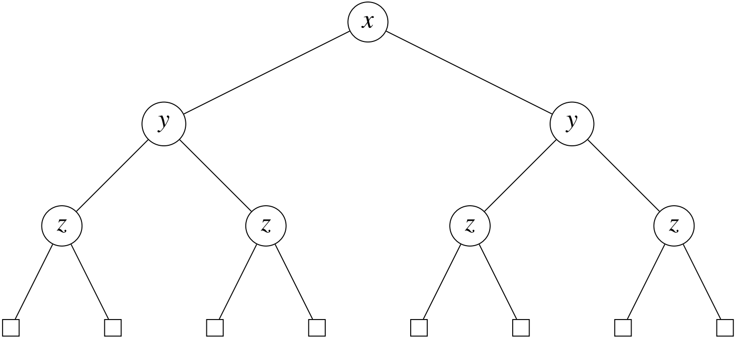

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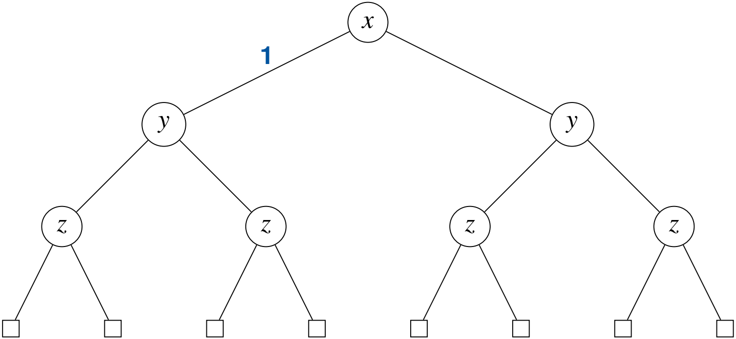

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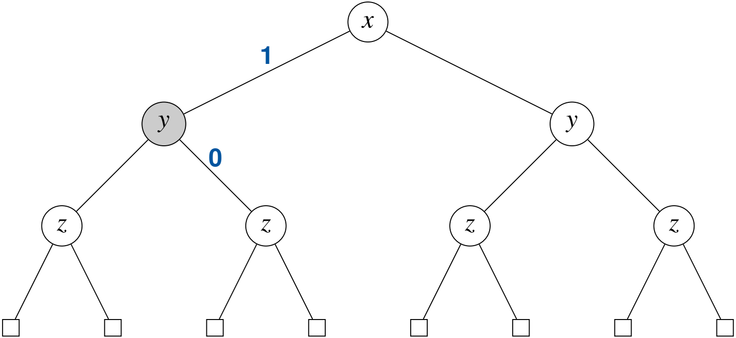

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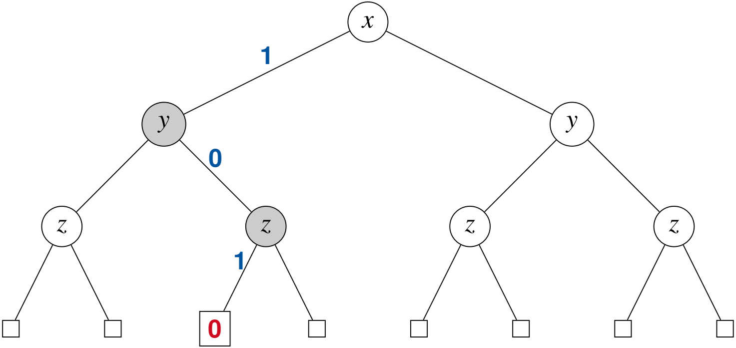

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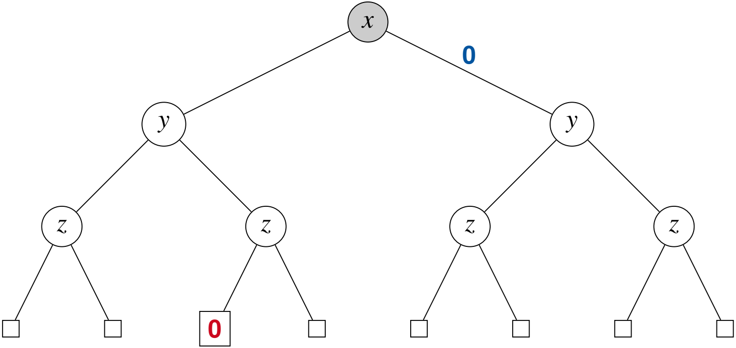

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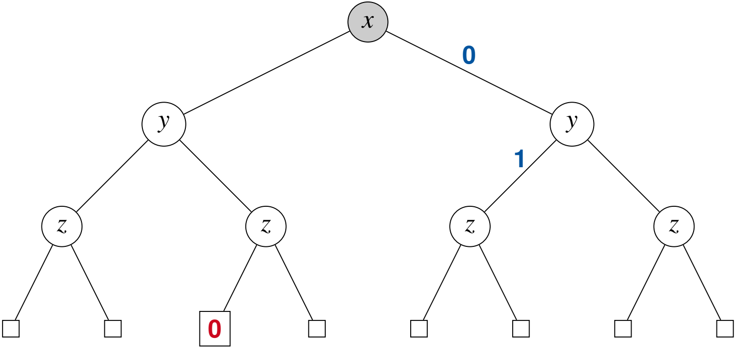

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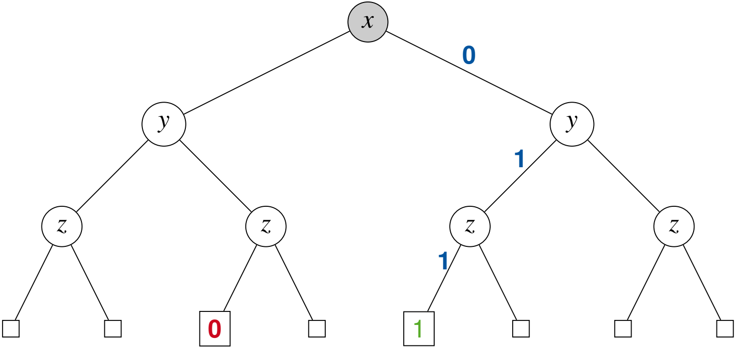

DPLL: Exploration with propagation

$$\underbrace{(\neg x \lor y \lor z)}_{c_1} \land \underbrace{(y \lor \neg z)}_{c_2} \land \underbrace{(\neg x \lor \neg y)}_{c_3}$$

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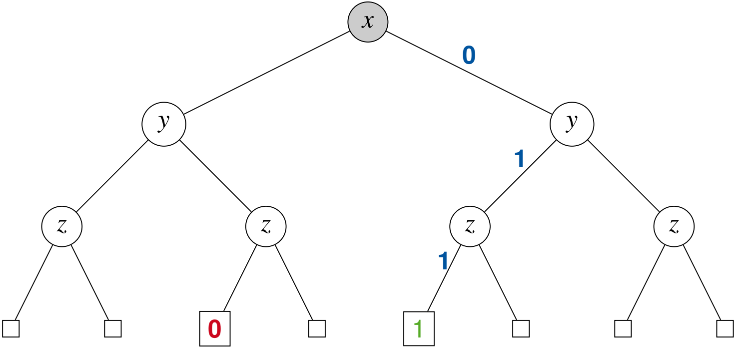

Efficient propagation with the watched literal scheme.

The DPLL+CDCL idea [Davis et al., '60/61] [Marques-Silva et al., '96]

Exploration:	$\mathbb{B}$-decision
Look-ahead:	$\mathbb{B}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution

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$\mathbb{B}$-propagate -

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$\mathbb{B}$-propagate -

$\mathbb{B}$-decision a = *false*

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$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*
$\mathbb{B}$-propagate	

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$\mathbb{B}$-decision	a = *false*
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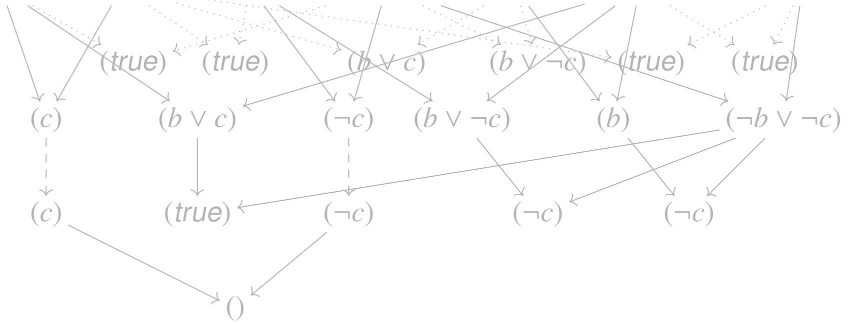
$\mathbb{B}$-conflict resolution

$$\frac{(\neg b \vee \neg c)\ (\neg b \vee c)}{(\neg b)}$$

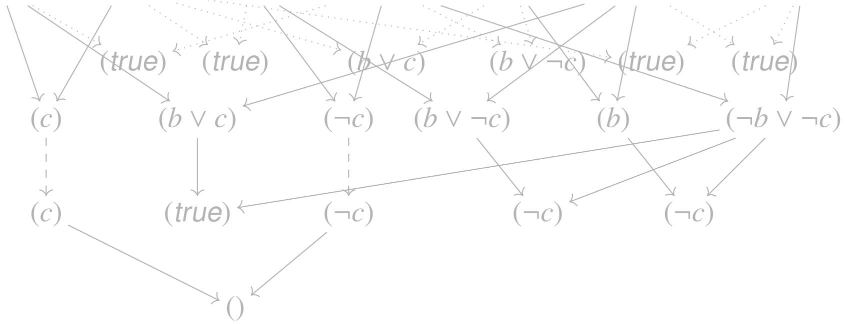
Resolution example revisited

$$(a \lor c) \land (\neg a \lor c) \land (a \lor \neg c) \land (\neg a \lor \neg c) \land (a \lor b) \land (\neg a \lor b) \land (a \lor \neg b \lor \neg c)$$



Resolution example revisited

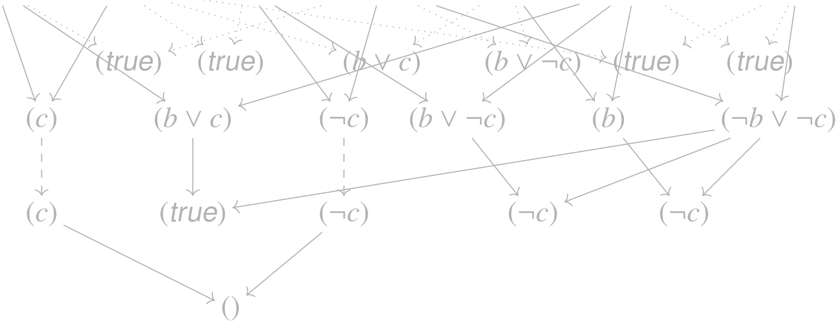
$$(a \vee c) \wedge (\neg a \vee c) \wedge (a \vee \neg c) \wedge (\neg a \vee \neg c) \wedge (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b \vee \neg c)$$



$\mathbb{B}$-propagate -

Resolution example revisited

$(a \vee c) \wedge (\neg a \vee c) \wedge (a \vee \neg c) \wedge (\neg a \vee \neg c) \wedge (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b \vee \neg c)$

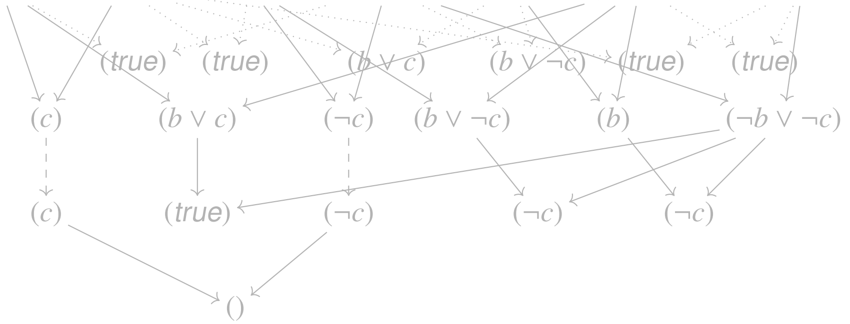


$\mathbb{B}$-propagate -

$\mathbb{B}$-decide $\neg c$

Resolution example revisited

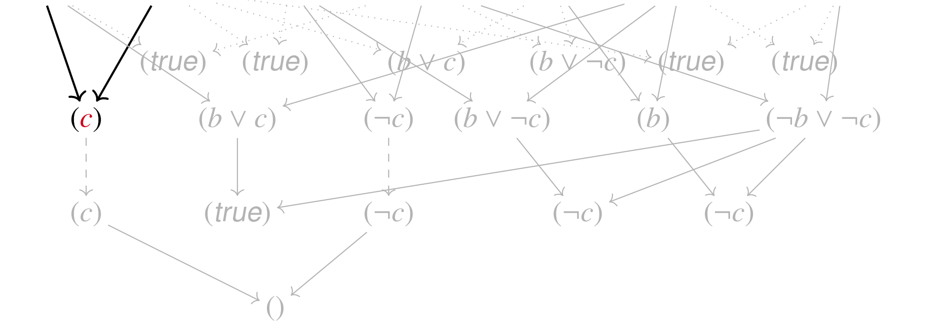
$(a \vee c) \wedge (\neg a \vee c) \wedge (a \vee \neg c) \wedge (\neg a \vee \neg c) \wedge (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b \vee \neg c)$



$\mathbb{B}$-propagate	-
$\mathbb{B}$-decide	$\neg c$
$\mathbb{B}$-propagate	a

Resolution example revisited

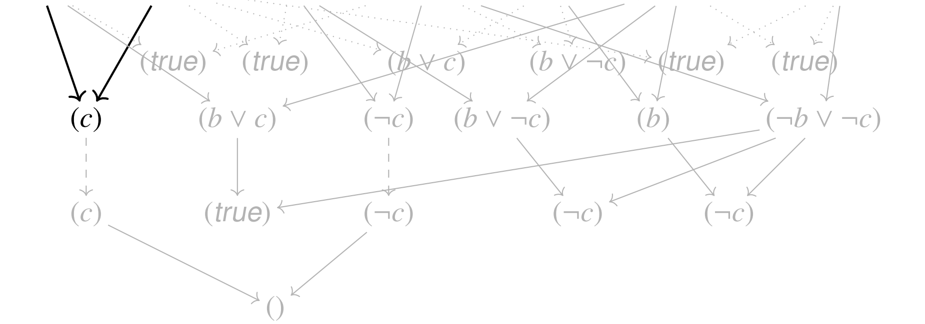
$(a \vee c) \wedge (\neg a \vee c) \wedge (a \vee \neg c) \wedge (\neg a \vee \neg c) \wedge (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b \vee \neg c)$



$\mathbb{B}$-propagate	-
$\mathbb{B}$-decide	$\neg c$
$\mathbb{B}$-propagate	a
$\mathbb{B}$-conflict resolution	$\dfrac{(\neg a \vee c)\ (a \vee c)}{(c)}$

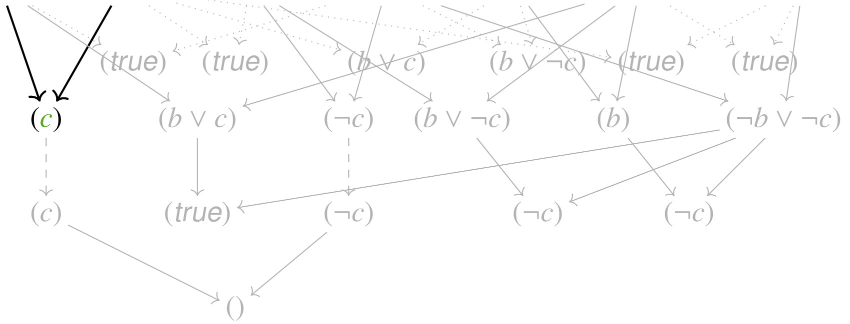
Resolution example revisited

$$(a \vee c) \wedge (\neg a \vee c) \wedge (a \vee \neg c) \wedge (\neg a \vee \neg c) \wedge (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b \vee \neg c)$$



Resolution example revisited

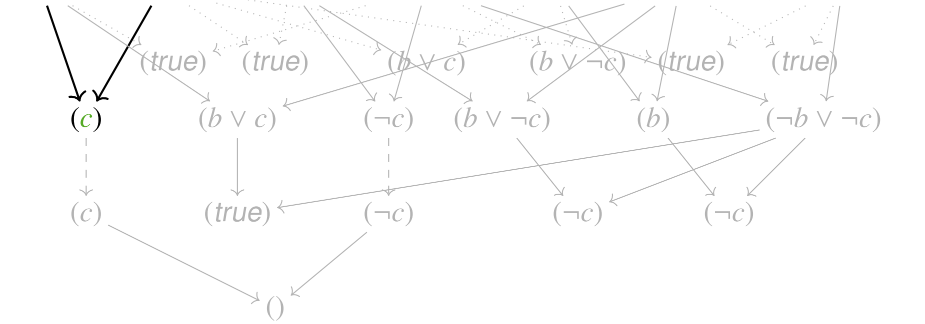
$(a \vee c) \wedge (\neg a \vee c) \wedge (a \vee \neg c) \wedge (\neg a \vee \neg c) \wedge (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b \vee \neg c)$



$\mathbb{B}$-propagate c

Resolution example revisited

$(a \vee c) \wedge (\neg a \vee c) \wedge (a \vee \neg c) \wedge (\neg a \vee \neg c) \wedge (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b \vee \neg c)$

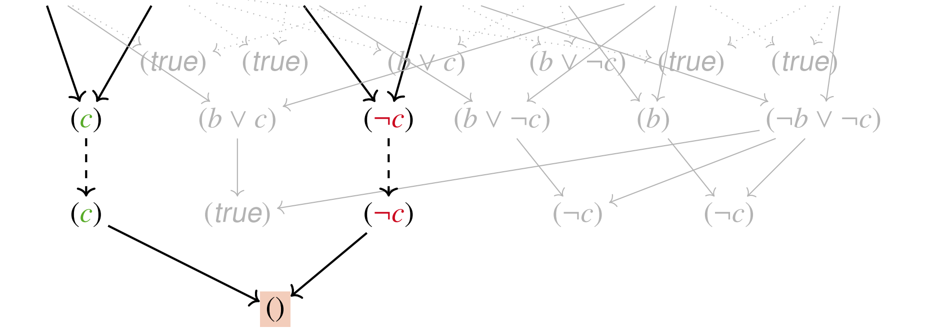


$\mathbb{B}$-propagate $\quad c$

a

Resolution example revisited

$$(a \vee c) \wedge (\neg a \vee c) \wedge (a \vee \neg c) \wedge (\neg a \vee \neg c) \wedge (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b \vee \neg c)$$



$\mathbb{B}$-propagate $\quad c$

a

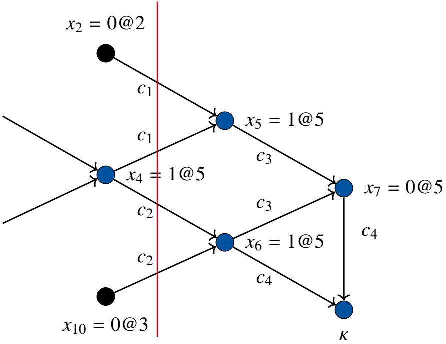
$\mathbb{B}$-conflict resolution

$$\cfrac{\cfrac{(\neg a \vee \neg c)\ (a \vee \neg c)}{(\neg c)} \quad (c)}{()}$$

Conflict clauses and (binary) resolution

- Consider the following example:

$c_1 = (\neg x_4 \vee x_2 \vee x_5)$
$c_2 = (\neg x_4 \vee x_{10} \vee x_6)$
$c_3 = (\neg x_5 \vee \neg x_6 \vee \neg x_7)$
$c_4 = (\neg x_6 \vee x_7)$
$\vdots \quad \vdots$

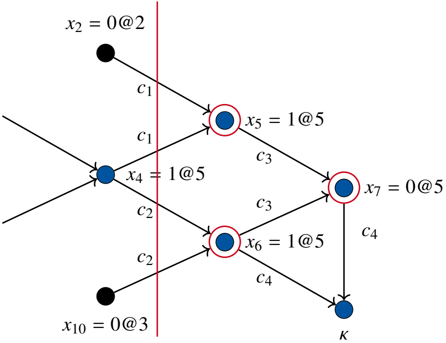


- Asserting conflict clause: $c_5 : (x_2 \vee \neg x_4 \vee x_{10})$

Conflict clauses and (binary) resolution

- Assigment order: x_4, x_5, x_6, x_7 Conflict clause: $c_5 : (x_2 \vee \neg x_4 \vee x_{10})$

$c_1 = (\neg x_4 \vee x_2 \vee x_5)$
$c_2 = (\neg x_4 \vee x_{10} \vee x_6)$
$c_3 = (\neg x_5 \vee \neg x_6 \vee \neg x_7)$
$c_4 = (\neg x_6 \vee x_7)$
$\vdots \qquad \vdots$



- Starting with the conflicting clause, apply resolution with the antecedent of the last assigned literal, until we get an asserting clause:
  - T1 = Res$(c_4, c_3, x_7) = (\neg x_5 \vee \neg x_6)$
  - T2 = Res(T1,$c_2, x_6) = (\neg x_4 \vee \neg x_5 \vee x_{10})$
  - T3 = Res(T2,$c_1, x_5) = (x_2 \vee \neg x_4 \vee x_{10})$

Unsatisfiable core

Definition

An unsatisfiable core of an unsatisfiable CNF formula is an unsatisfiable subset of the original set of clauses.

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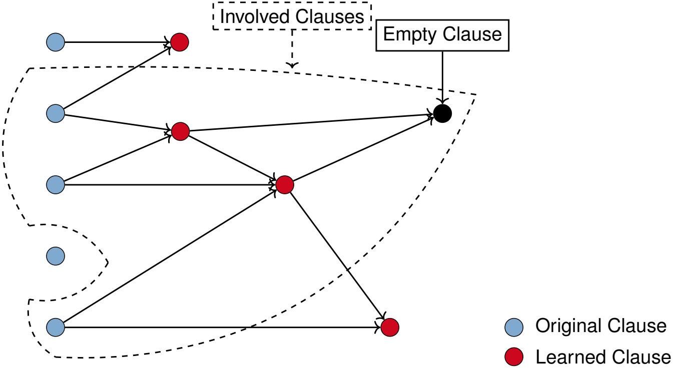
- The set of those original clauses that were used for resolution in conflict analysis during SAT-solving (inclusively the last conflict at decision level 0) gives us an unsatisfiable core which is in general much smaller.
- However, this unsatifiable core is still not always minimal (i.e., we can remove clauses from it still having an unsatisfiable core).

The resolution graph

A resolution graph gives us more information to get a minimal unsatisfiable core.



Termination

Theorem

It is never the case that the solver enters decision level dl again with the same partial assignment.

Termination

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It is never the case that the solver enters decision level dl again with the same partial assignment.

Proof.

Define a partial order on partial assignments: $\alpha < \beta$ iff either α is an extension of β or α has more assignments at the smallest decision level at that α and β do not agree.
BCP decreases the order, conflict-driven backtracking also. Since the order always decreases during the search, the theorem holds. □

Decision heuristics: VSIDS

- VSIDS (variable state independent decaying sum)
- Gives priority to variables involved in recent conflicts.
- “Involved” can have different definitions. We take those variables that occur in clauses used for conflict resolution.

Decision heuristics: VSIDS

- VSIDS (variable state independent decaying sum)
- Gives priority to variables involved in recent conflicts.
- "Involved" can have different definitions. We take those variables that occur in clauses used for conflict resolution.

1. Each variable has a counter initialized to 0.
2. We define an increment value (e.g., 1).
3. When a conflict occurs, we increase the counter of each variable, that occurs in at least one clause used for conflict resolution, by the increment value.
   Afterwards we increase the increment value (e.g., by 1).
4. For decisions, the unassigned variable with the highest counter is chosen.
5. Periodically, all the counters and the increment value are divided by a constant.

Decision heuristics: VSIDS

VSIDS is a ’quasi-static’ strategy:

- static because it doesn’t depend on current assignment
- dynamic because it gradually changes. Variables that appear in recent conflicts have higher priority.

  This strategy is a conflict-driven decision strategy.

”...employing this strategy dramatically (i.e., an order of magnitude) improved performance...”

Contents

Example 1: Seminar topic assignment

- n participants
- n topics
- Set of preferences $E \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$

  $(p, t) \in E$ means: participant p would take topic t

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- Q: Can we assign to each participant a topic which he/she is willing to take?

Example 1: Propositional encoding

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Each participant is willing to take his/her assigned topic:

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Each participant is willing to take his/her assigned topic:

$$\bigwedge_{p=1}^{n} \bigwedge_{(p,t) \notin E} \neg x_{p,t}$$

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Each topic is assigned to at most one participant:

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Example 2: Placement of wedding guests

- Three chairs in a row: $1, 2, 3$
- We need to place Aunt, Sister and Father.
- Constraints:
  - Aunt doesn't want to sit near Father
  - Aunt doesn't want to sit in the left chair
  - Sister doesn't want to sit to the right of Father

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- We need to place Aunt, Sister and Father.
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  - Aunt doesn't want to sit near Father
  - Aunt doesn't want to sit in the left chair
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- Q: Can we satisfy these constraints?

Example 2: Propositional encoding

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- Notation:

Example 2: Propositional encoding

- Notation: Aunt = 1, Sister = 2, Father = 3
  Left chair = 1, Middle chair = 2, Right chair = 3

  Introduce a propositional variable for each pair (person, chair):

  $x_{p,c}$ = "person p is sited in chair c" for $1 \leq p, c \leq 3$

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  Aunt doesn't want to sit near Father:

$$((x_{1,1} \vee x_{1,3}) \rightarrow \neg x_{3,2}) \wedge (x_{1,2} \rightarrow (\neg x_{3,1} \wedge \neg x_{3,3}))$$

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$$\neg x_{1,1}$$

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$$(x_{3,1} \rightarrow \neg x_{2,2}) \wedge (x_{3,2} \rightarrow \neg x_{2,3})$$

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Example 2: Propositional encoding

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Example 3: Assignment of frequencies

- n radio stations
- For each station assign one of k transmission frequencies, $k < n$.
- E – set of pairs of stations, that are too close to have the same frequency.

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- n radio stations
- For each station assign one of k transmission frequencies, $k < n$.
- E – set of pairs of stations, that are too close to have the same frequency.
- Q: Can we assign to each station one frequency, such that no station pairs from E have the same frequency?

Example 3: Propositional encoding

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  $x_{s,f}$ = "station s is assigned frequency f" for $1 \leq s \leq n$, $1 \leq f \leq k$

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  Close stations are not assigned the same frequency:

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  Close stations are not assigned the same frequency:

  For each $(s1, s2) \in E$,

$$\bigwedge_{f=1}^{k} \left(\neg x_{s1,f} \vee \neg x_{s2,f} \right)$$

Contents

You need to have installed...

- Python
- Z3

  `https://github.com/exercism/z3/blob/main/docs/INSTALLATION.md`

SAT encodings

- Suppose we can solve the satisfiability problem... how can this help us?

SAT encodings

- Suppose we can solve the satisfiability problem... how can this help us?

- There are numerous problems in the industry that are solved via the satisfiability problem of propositional logic
  - Logistics
  - Planning
  - Electronic Design Automation industry
  - Cryptography
  - ...

DIMACS input syntax for SAT solvers

The DIMACS format for SAT solvers has three types of lines:

- header: “p cnf n m” in which
  - n denotes the highest variable index and
  - m the number of clauses.
- clauses: a sequence of integers ending with “0”
- comments: any line starting with “c “

Example:

		c example
		p cnf 2 4
$(a \vee b)$	$\wedge$	$1 \quad 2 \quad 0$
$(\neg a \vee b)$	$\wedge$	$-1 \quad 2 \quad 0$
$(a \vee \neg b)$	$\wedge$	$1 \quad -2 \quad 0$
$(\neg a \vee \neg b)$	$\wedge$	$-1 \quad -2 \quad 0$

Example 2 (wedding): DIMACS format

Notation: Aunt = 1, Sister = 2, Father = 3
Left chair = 1, Middle chair = 2, Right chair = 3
$x_{p,c}$ = "person p is sited in chair c" for $1 \leq p, c \leq 3$

Constraints:

(1) $((x_{1,1} \vee x_{1,3}) \rightarrow \neg x_{3,2}) \wedge (x_{1,2} \rightarrow (\neg x_{3,1} \wedge \neg x_{3,3}))$

(2) $\neg x_{1,1}$

(3) $(x_{3,1} \rightarrow \neg x_{2,2}) \wedge (x_{3,2} \rightarrow \neg x_{2,3})$

(4) $\bigwedge_{p=1}^{3} \bigvee_{c=1}^{3} x_{p,c}$

(5) $\bigwedge_{p_1=1}^{3} \bigwedge_{p_2=p_1+1}^{3} \bigwedge_{c=1}^{3} (\neg x_{p_1,c} \vee \neg x_{p_2,c})$

	c example
	p cnf 2 4
$(a \vee b) \quad \wedge$	$1 \quad 2 \qquad 0$
$\ldots$	$\ldots$

Example 3 (frequencies): DIMACS format

$$
\begin{array}{ll}
(1) & \bigwedge_{s=1}^{n} \left(\bigvee_{f=1}^{k} x_{s,f} \right) \\
(2) & \bigwedge_{s=1}^{n} \bigwedge_{f_1=1}^{k-1} \bigwedge_{f_2=f_1+1}^{k} \left(\neg x_{s,f_1} \vee \neg x_{s,f_2} \right) \\
(3) & \forall (s_1, s_2) \in E.\ \bigwedge_{f=1}^{k} \left(\neg x_{s_1,f} \vee \neg x_{s_2,f} \right)
\end{array}
$$

Example 3 (frequencies): DIMACS format

$$\begin{array}{ll}(1) & \bigwedge_{s=1}^{n}\left(\bigvee_{f=1}^{k} x_{s,f}\right)\\ (2) & \bigwedge_{s=1}^{n}\bigwedge_{f_1=1}^{k-1}\bigwedge_{f_2=f_1+1}^{k}\left(\neg x_{s,f_1} \vee \neg x_{s,f_2}\right)\\ (3) & \forall (s_1,s_2)\in E.\ \bigwedge_{f=1}^{k}\left(\neg x_{s_1,f} \vee \neg x_{s_2,f}\right)\end{array}$$

Assume that n^2 ($n \in \mathbb{N}_{>0}$) stations are arranged in a grid with the coordinates (i,j), $1 \leq i,j \leq n$, and

$$\begin{array}{rcl} E & = & \{((i,j),(i+1,j)) \mid 1 \leq i \leq n-1 \wedge 1 \leq j \leq n\} \cup \\ & & \{((i,j),(i,j+1)) \mid 1 \leq i \leq n \wedge 1 \leq j \leq n-1\}. \end{array}$$

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$$
\begin{array}{rcl}
E & = & \{((i,j),(i+1,j)) \,|\, 1 \leq i \leq n-1 \wedge 1 \leq j \leq n\} \cup \\
& & \{((i,j),(i,j+1)) \,|\, 1 \leq i \leq n \wedge 1 \leq j \leq n-1\}.
\end{array}
$$

Write a Python program that writes for an input n the **DIMACS** encoding for $k = 1, \ldots, n$ into an external file, and check them (by manually calling z3 on them) to identify the minimal k necessary for a solution.

Example 3: DIMACS

```
import argparse
import sys
try:
    parser = argparse.ArgumentParser()
    parser.add_argument("n", help="number of stations", type=int)
    args = parser.parse_args()
    n = args.n
except:
    e = sys.exc_info()[0]
    print(e)

for k in range(n):
    names = []
    for i in range(n):
        names_i = []
        for j in range(k+1):
            name = str(i*(k+1)+j+1)
            names_i.append(name)
        names.append(names_i)
    clauses = ""
    counter = 0
    for i in range(n):
        for j in range(k+1):
            clauses += names[i][j] + " "
        clauses += "0\n"
        counter += 1
    file = open("frequencies" + str(k+1) + ".dimacs","w")
    file.write("p cnf " + str(n*(k+1)) + " " + str(counter) + "\n")
    file.write(clauses)
    file.close()
```

Solving propositional logic with SMT solvers

- SMT-LIB format:
  https://microsoft.github.io/z3guide/docs/logic/propositional-logic
- Python interface:
  https://ericpony.github.io/z3py-tutorial/guide-examples.htm
- Both:
  https://cvc5.github.io/tutorials/beginners/

SMT-LIB2 format

Boolean SMT-LIB example

```
(set-logic QF_UF)
(declare-const p Bool)
(assert (and p (not p)))
(check-sat)
```

Example 3 (frequencies): SMT-LIB format

$$
\begin{array}{ll}
(1) & \bigwedge_{s=1}^{n} \left(\bigvee_{f=1}^{k} x_{s,f} \right) \\
(2) & \bigwedge_{s=1}^{n} \bigwedge_{f_1=1}^{k-1} \bigwedge_{f_2=f_1+1}^{k} \left(\neg x_{s,f_1} \vee \neg x_{s,f_2} \right) \\
(3) & \forall (s_1, s_2) \in E.\ \bigwedge_{f=1}^{k} \left(\neg x_{s_1,f} \vee \neg x_{s_2,f} \right)
\end{array}
$$

Assume that n^2 ($n \in \mathbb{N}_{>0}$) stations are arranged in a grid with the coordinates (i,j), $1 \leq i,j \leq n$, and

$$
\begin{array}{rcl}
E & = & \{((i,j),(i{+}1,j)) \,|\, 1 \leq i \leq n{-}1 \wedge 1 \leq j \leq n\} \cup \\
 & & \{((i,j),(i,j{+}1)) \,|\, 1 \leq i \leq n \wedge 1 \leq j \leq n{-}1\}\,.
\end{array}
$$

Example 3 (frequencies): SMT-LIB format

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(1) & \bigwedge_{s=1}^{n} \left(\bigvee_{f=1}^{k} x_{s,f} \right) \\
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\end{array}
$$

Assume that n^2 ($n \in \mathbb{N}_{>0}$) stations are arranged in a grid with the coordinates (i, j), $1 \leq i, j \leq n$, and

$$
\begin{aligned}
E \quad = \quad & \{((i,j),(i+1,j)) \mid 1 \leq i \leq n-1 \wedge 1 \leq j \leq n\} \cup \\
& \{((i,j),(i,j+1)) \mid 1 \leq i \leq n \wedge 1 \leq j \leq n-1\}\,.
\end{aligned}
$$

Write a Python program that writes for an input n the **SMT-LIB2** encoding for $k = 1, \ldots, n$ into an external file, and check them (by manually calling z3 on them) to identify the minimal k necessary for a solution.

Example 3: SMT-LIB2

```
import argparse
import sys
try:
     parser = argparse.ArgumentParser()
     parser.add_argument("n", help="number of stations", type=int)
     args = parser.parse_args()
     n = args.n
except:
    e = sys.exc_info()[0]
    print(e)

names = []
for i in range(n):
    names_i = []
    for j in range(n):
        name = "a_" + str(i+1) + "_" + str(j+1);
        names_i.append(name)
    names.append(names_i)
for k in range(n):
    file = open("frequencies" + str(k+1) + ".smt2","w")
    file.write("(set-logic QF_UF)\n")
    for i in range(n):
        for j in range(k+1):
            file.write("(declare-const " + names[i][j] + " Bool)\n")
    for i in range(n):
        file.write("(assert (or")
        for j in range(k+1):
            file.write(" " + names[i][j])
        file.write("))\n")
    file.write("(check-sat)\n")
    file.write("(exit)\n")
    file.close()
```

Solving propositional logic with SMT solvers

- SMT-LIB format:
  https://microsoft.github.io/z3guide/docs/logic/propositional-logic
- Python interface:
  https://ericpony.github.io/z3py-tutorial/guide-examples.htm
- Both:
  https://cvc5.github.io/tutorials/beginners/

Example 3: Python API

$$
\begin{array}{ll}
(1) & \bigwedge_{s=1}^{n} \left(\bigvee_{f=1}^{k} x_{s,f} \right) \\
(2) & \bigwedge_{s=1}^{n} \bigwedge_{f_1=1}^{k-1} \bigwedge_{f_2=f_1+1}^{k} \left(\neg x_{s,f_1} \vee \neg x_{s,f_2} \right) \\
(3) & \forall (s_1, s_2) \in E.\ \bigwedge_{f=1}^{k} \left(\neg x_{s_1,f} \vee \neg x_{s_2,f} \right)
\end{array}
$$

Assume that n^2 ($n \in \mathbb{N}_{>0}$) stations are arranged in a grid with the coordinates (i,j), $1 \leq i,j \leq n$, and

$$
\begin{array}{rcl}
E & = & \{((i,j),(i{+}1,j)) \mid 1 \leq i \leq n{-}1 \wedge 1 \leq j \leq n\} \cup \\
 & & \{((i,j),(i,j{+}1)) \mid 1 \leq i \leq n \wedge 1 \leq j \leq n{-}1\}\,.
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(3) & \forall (s_1, s_2) \in E.\ \bigwedge_{f=1}^{k} \left(\neg x_{s_1,f} \vee \neg x_{s_2,f} \right)
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\end{array}
$$

Write a Python program that uses for an input n the **Python API** of z3 to find the minimal k necessary for a solution.

Example 3: Python API

```
from z3 import *
import argparse
import sys
try:
     parser = argparse.ArgumentParser()
     parser.add_argument("n", help="number of stations", type=int)
     args = parser.parse_args()
     n = args.n
except:
    e = sys.exc_info()[0]
    print(e)

names = []
for i in range(n):
    names_i = []
    for j in range(n):
        name = "a_" + str(i+1) + "_" + str(j+1);
        names_i.append(Bool(name))
    names.append(names_i)
s = Solver()
for k in range(n):
    s.push()
    for i in range(n):
        params = []
        for j in range(k+1):
            params.append(names[i][j])
        s.add(Or(params))
    print(s)
    print(s.check())
    s.pop()
```

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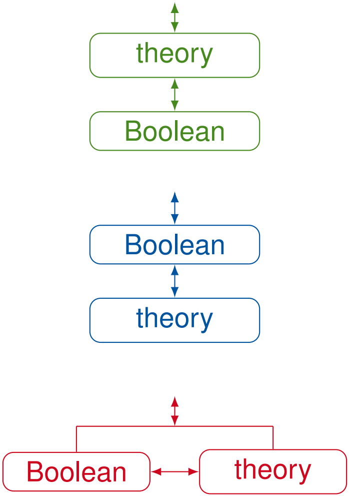
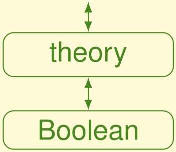
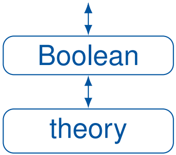


Three SMT solving approaches

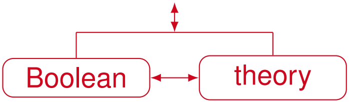
Eager SMT solving



Lazy SMT solving



Model-constructing satisfiability calculus (MCSAT)



Eager example [Bryant and Velev, 2000]

$$\varphi^E \;=\; x_1 = x_2 \;\wedge\; x_2 = x_3 \;\wedge\; x_1 \neq x_3$$

Eager example [Bryant and Velev, 2000]

$$\varphi^E \;=\; x_1 = x_2 \;\wedge\; x_2 = x_3 \;\wedge\; x_1 \neq x_3$$

$$\varphi^{prop} \;:=$$

φ^E *is satisfiable* *iff* φ^{prop} *is satisfiable*

Eager example [Bryant and Velev, 2000]

$$\varphi^E = x_1 = x_2 \wedge x_2 = x_3 \wedge x_1 \neq x_3$$

$$\varphi^{prop} := \underbrace{e_1 \wedge e_2 \wedge \neg e_3}_{\textit{Boolean abstraction}} \wedge$$

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φ^E *is satisfiable* *iff* φ^{prop} *is satisfiable*

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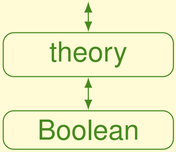
φ^E *is satisfiable* *iff* φ^{prop} *is satisfiable*

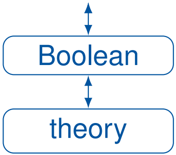
Similar approaches are available for uninterpreted functions, bit-vector arithmetic ("bit-blasting"), floating-point arithmetic and others.

Three SMT solving approaches

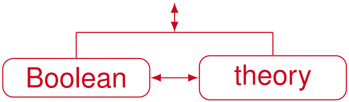
Eager SMT solving



Lazy SMT solving

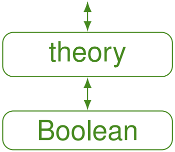


Model-constructing satisfiability calculus (MCSAT)

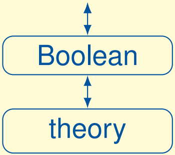


Three SMT solving approaches

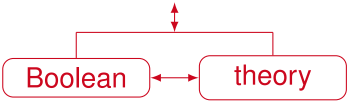
Eager SMT solving



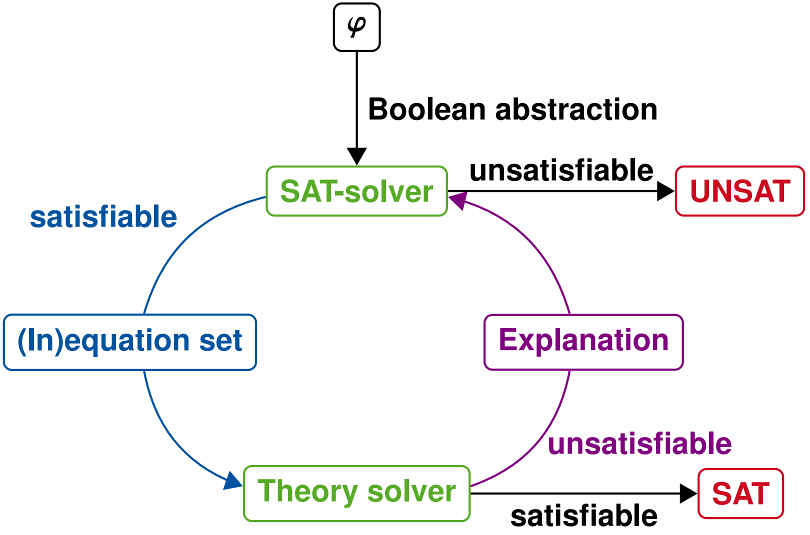
Lazy SMT solving



Model-constructing satisfiability calculus (MCSAT)



Full lazy SMT solving



Boolean abstraction

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$

$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$

$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9}$$

Boolean abstraction

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$

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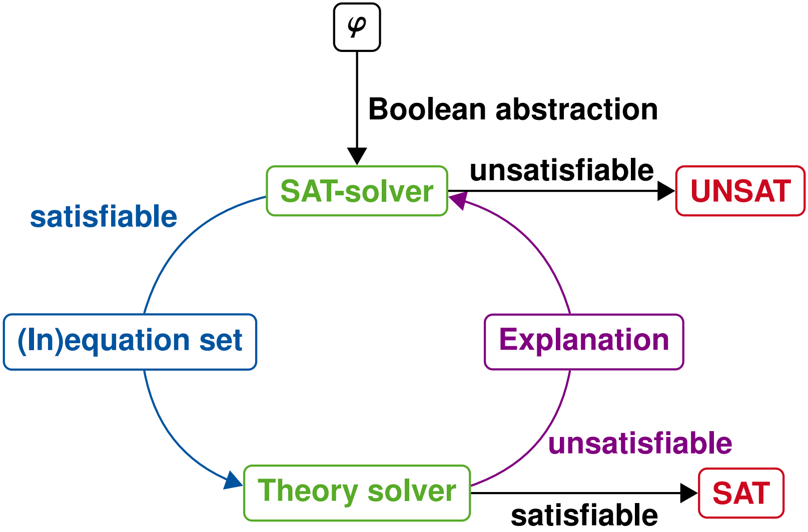
$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Full lazy SMT solving



SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0$:

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

SAT solving

$$(a_1 \lor a_2 \lor a_3) \land a_4 \land (a_5 \lor a_6) \land a_7 \land a_8 \land a_9$$

Assume a fixed variable order: $a_1, \ldots, a_9$
Assignment to decision variables: false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 :$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 :$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

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$DL2 : a_2 : 0, a_3 : 1$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

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Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
$DL3 :$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
Assignment to decision variables:	false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
$DL3 : a_5 : 0$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order:	$a_1, \ldots, a_9$
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SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

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$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
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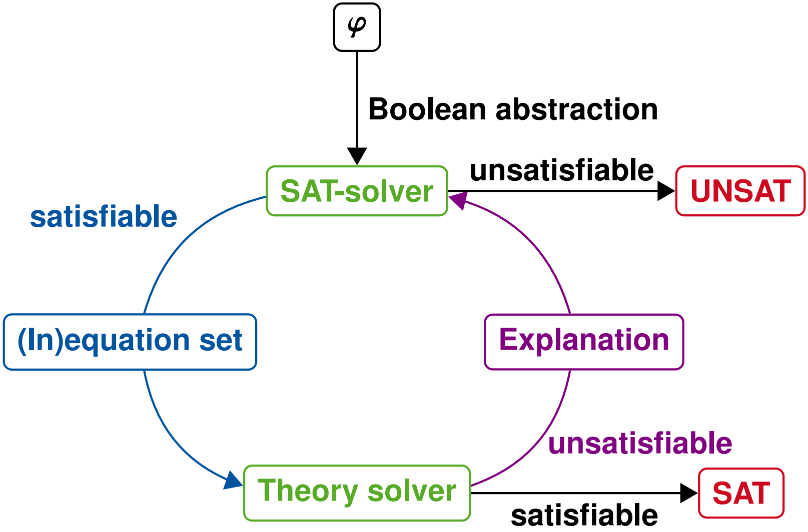
Solution found for the Boolean abstraction.

Full lazy SMT solving



Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$ $\quad DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$ $\quad DL3 : a_5 : 0, a_6 : 1$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$ $DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$ $DL3 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_3, a_6$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$ $\quad DL1 : a_1 : 0$
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True theory constraints: $a_4, a_7, a_8, a_9, a_3, a_6$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$

$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$

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$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$ $DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$ $DL3 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_3, a_6$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
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Encoding:

$a_4 : p_1 + p_2 + p_3 \geq 100$ $a_7 : p_3 \geq 10$ $a_8 : p_1 + 2p_2 + 5p_3 \leq 180$
$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$ $a_3 : p_3 = 0$ $a_6 : p_2 \geq 5$

Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

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$a_6 : p_2 \geq 5$

No.

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

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$a_3 : p_3 = 0$

$a_6 : p_2 \geq 5$

No.

Reason:

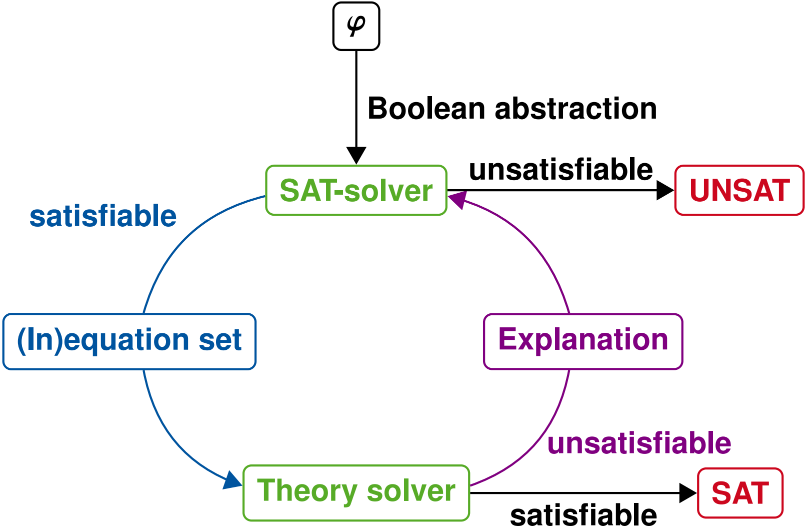
Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$
$a_7 : p_3 \geq 10$
$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$
$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$
$a_3 : p_3 = 0$
$a_6 : p_2 \geq 5$

No.
Reason: $\underbrace{p_3 = 0}_{a_3} \wedge \underbrace{p_3 \geq 10}_{a_7}$ are conflicting.

Full lazy SMT solving



SAT solving

Add clause $(\neg a_3 \vee \neg a_7)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
$DL3 : a_5 : 0, a_6 : 1$

Add clause $(\neg a_3 \vee \neg a_7)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
$DL3 : a_5 : 0, a_6 : 1$

Conflict resolution is simple, since the new clause is already an asserting one.

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 :$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
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$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 :$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

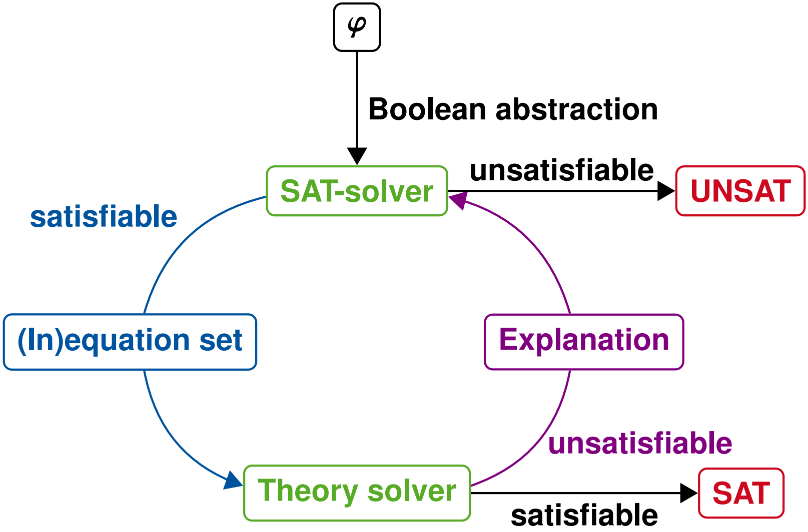
$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

Solution found for the Boolean abstraction.

Full lazy SMT solving



Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_6$

Theory solving

$DL0: a_4:1, a_7:1, a_8:1, a_9:1, a_3:0 \quad DL1: a_1:0, a_2:1$
$DL2: a_5:0, a_6:1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_6$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_6$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7)$$

Encoding:

$a_4 : p_1 + p_2 + p_3 \geq 100 \quad a_7 : p_3 \geq 10 \quad a_8 : p_1 + 2p_2 + 5p_3 \leq 180$
$a_9 : 3p_1 + 2p_2 + p_3 \leq 300 \quad a_2 : p_2 = 0 \quad a_6 : p_2 \geq 5$

Theory solving

Is the conjunction of the following constraints satisfiable?

a_4 : $p_1 + p_2 + p_3 \geq 100$

a_7 : $p_3 \geq 10$

a_8 : $p_1 + 2p_2 + 5p_3 \leq 180$

a_9 : $3p_1 + 2p_2 + p_3 \leq 300$

a_2 : $p_2 = 0$

a_6 : $p_2 \geq 5$

Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_2 : p_2 = 0$

$a_6 : p_2 \geq 5$

No.

Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_2 : p_2 = 0$

$a_6 : p_2 \geq 5$

No.

Reason:

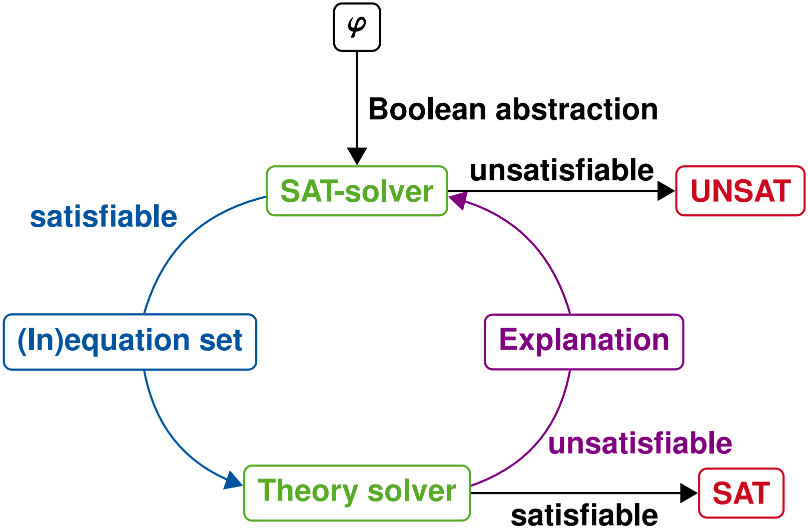
Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$
$a_7 : p_3 \geq 10$
$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$
$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$
$a_2 : p_2 = 0$
$a_6 : p_2 \geq 5$

No.
Reason: $\underbrace{p_2 = 0}_{a_2} \wedge \underbrace{p_2 \geq 5}_{a_6}$ are conflicting.

Full lazy SMT solving



Add clause $(\neg a_2 \vee \neg a_6)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

Add clause $(\neg a_2 \vee \neg a_6)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

Conflict resolution is simple, since the new clause is already an asserting one.

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$

SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1, a_6 : 0$

SAT solving

$$(a_1 \lor a_2 \lor a_3) \land a_4 \land (a_5 \lor a_6) \land a_7 \land a_8 \land a_9 \land (\neg a_3 \lor \neg a_7) \land$$
$$(\neg a_2 \lor \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

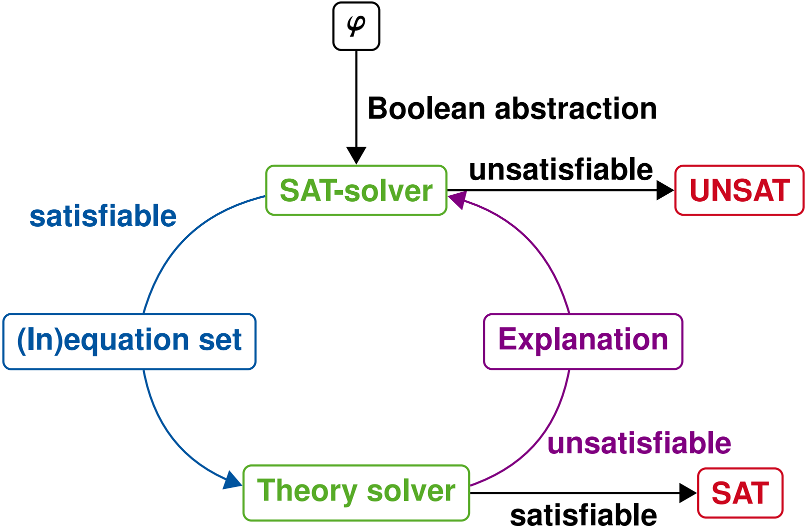
SAT solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

Solution found for the Boolean abstraction.

Full lazy SMT solving



Theory solving

$DL0: a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1: a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$

$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$

$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$

$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$

$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

Encoding:

$a_4 : p_1 + p_2 + p_3 \geq 100 \quad a_7 : p_3 \geq 10 \quad a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300 \quad a_2 : p_2 = 0 \quad a_5 : p_1 \geq 5$

Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_2 : p_2 = 0$

$a_5 : p_1 \geq 5$

Theory solving

Is the conjunction of the following constraints satisfiable?

$a_4 : p_1 + p_2 + p_3 \geq 100$

$a_7 : p_3 \geq 10$

$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$

$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$

$a_2 : p_2 = 0$

$a_5 : p_1 \geq 5$

Yes.

Is the conjunction of the following constraints satisfiable?

a_4 : $p_1 + p_2 + p_3 \geq 100$
a_7 : $p_3 \geq 10$
a_8 : $p_1 + 2p_2 + 5p_3 \leq 180$
a_9 : $3p_1 + 2p_2 + p_3 \leq 300$
a_2 : $p_2 = 0$
a_5 : $p_1 \geq 5$

Yes. E.g.,

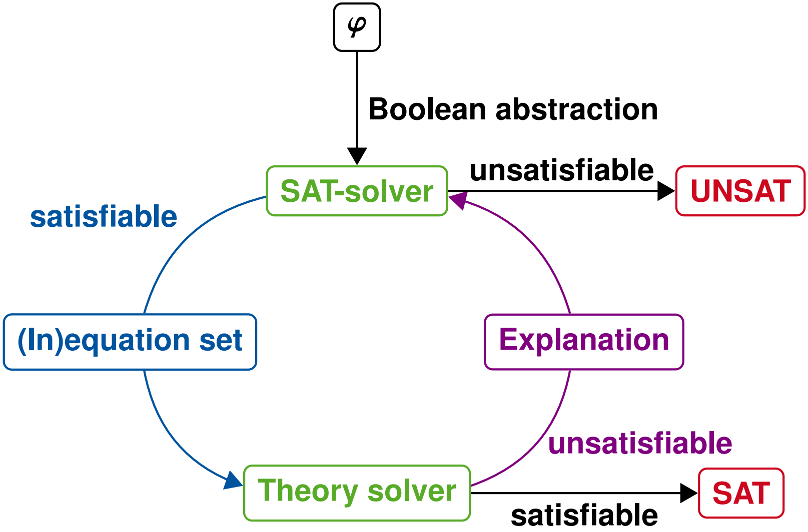
Theory solving

Is the conjunction of the following constraints satisfiable?

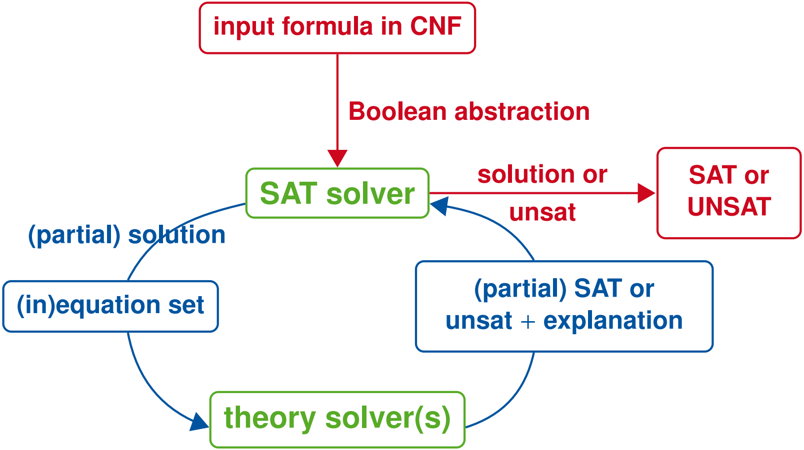
a_4 : $p_1 + p_2 + p_3 \geq 100$
a_7 : $p_3 \geq 10$
a_8 : $p_1 + 2p_2 + 5p_3 \leq 180$
a_9 : $3p_1 + 2p_2 + p_3 \leq 300$
a_2 : $p_2 = 0$
a_5 : $p_1 \geq 5$

Yes. E.g., $p_1 = 90$, $p_2 = 0$, $p_3 = 10$ is a solution.

Full lazy SMT solving



Less lazy SMT solving

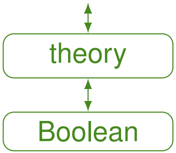


Requirements on the theory solver

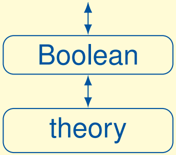
1. Incrementality: In less lazy solving we extend the set of constraints. The solver should make use of the previous satisfiability check for the check of the extended set.
2. (Preferably minimal) infeasible subsets: Compute a reason for unsatisfaction
3. Backtracking: The theory solver should be able to remove constraints in inverse chronological order.

Three SMT solving approaches

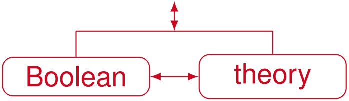
Eager SMT solving



Lazy SMT solving

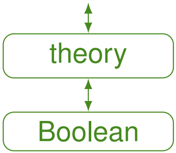


Model-constructing satisfiability calculus (MCSAT)

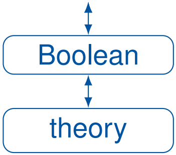


Three SMT solving approaches

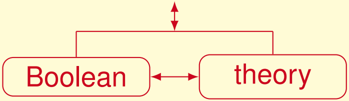
Eager SMT solving



Lazy SMT solving



Model-constructing satisfiability calculus (MCSAT)



The DPLL+CDCL idea [Davis et al., ’60/61] [Marques-Silva et al., ’96]

Exploration: $\mathbb{B}$-decision

Look-ahead: $\mathbb{B}$-propagation

Proof system: $\mathbb{B}$-conflict resolution

The DPLL+CDCL idea [Davis et al., '60/61] [Marques-Silva et al., '96]

Exploration: $\mathbb{B}$-decision

Look-ahead: $\mathbb{B}$-propagation

Proof system: $\mathbb{B}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

The DPLL+CDCL idea [Davis et al., '60/61] [Marques-Silva et al., '96]

Exploration: $\mathbb{B}$-decision

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Proof system: $\mathbb{B}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

$\mathbb{B}$-propagate -

The DPLL+CDCL idea [Davis et al., ’60/61] [Marques-Silva et al., ’96]

Exploration: $\mathbb{B}$-decision

Look-ahead: $\mathbb{B}$-propagation

Proof system: $\mathbb{B}$-conflict resolution

$$(a \lor b \lor c) \land (a \lor b \lor \neg c)$$

$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	$a = $ *false*

The DPLL+CDCL idea [Davis et al., '60/61] [Marques-Silva et al., '96]

Exploration: $\mathbb{B}$-decision

Look-ahead: $\mathbb{B}$-propagation

Proof system: $\mathbb{B}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

$\mathbb{B}$-propagate -

$\mathbb{B}$-decision $a =$ *false*

$\mathbb{B}$-propagate -

Exploration: $\mathbb{B}$-decision

Look-ahead: $\mathbb{B}$-propagation

Proof system: $\mathbb{B}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*
$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	b = *false*

The DPLL+CDCL idea [Davis et al., '60/61] [Marques-Silva et al., '96]

Exploration:	$\mathbb{B}$-decision
Look-ahead:	$\mathbb{B}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*
$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	b = *false*
$\mathbb{B}$-propagate	c = *true* $\lightning$

The DPLL+CDCL idea [Davis et al., ’60/61] [Marques-Silva et al., ’96]

Exploration: $\mathbb{B}$-decision

Look-ahead: $\mathbb{B}$-propagation

Proof system: $\mathbb{B}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*
$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	b = *false*
$\mathbb{B}$-propagate	c = *true* ⚡
$\mathbb{B}$-conflict resolution	$(a \vee b)$

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$$

$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*
$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	b = *false*
$\mathbb{B}$-propagate	c = *true* ⚡
$\mathbb{B}$-conflict resolution	$(a \vee b)$

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$ $\qquad \ldots\ x \cdot y^2 < 0\ \ldots$

$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*
$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	b = *false*
$\mathbb{B}$-propagate	c = *true* ↯
$\mathbb{B}$-conflict resolution	$(a \vee b)$

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c) \qquad \ldots\ x \cdot y^2 < 0\ \ldots$$

$\mathbb{B}$-propagate	-	$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*		
$\mathbb{B}$-propagate	-		
$\mathbb{B}$-decision	b = *false*		
$\mathbb{B}$-propagate	c = *true* ⚡		
$\mathbb{B}$-conflict resolution	$(a \vee b)$		

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$ $\quad \ldots\ x \cdot y^2 < 0\ \ldots$

$\mathbb{B}$-propagate	-	$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*	$\mathbb{B}$-decision	$x \cdot y^2 < 0$
$\mathbb{B}$-propagate	-		
$\mathbb{B}$-decision	b = *false*		
$\mathbb{B}$-propagate	c = *true* ↯		
$\mathbb{B}$-conflict resolution	$(a \vee b)$		

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$ $\quad\ldots\ x \cdot y^2 < 0\ \ldots$

$\mathbb{B}$-propagate	-	$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*	$\mathbb{B}$-decision	$x \cdot y^2 < 0$
$\mathbb{B}$-propagate	-	$\mathbb{T}$-propagate	$x \in (-\infty, \infty)$
$\mathbb{B}$-decision	b = *false*		
$\mathbb{B}$-propagate	c = *true* ↯		
$\mathbb{B}$-conflict resolution	$(a \vee b)$		

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$$(a \vee b \vee c) \wedge (a \vee b \vee \neg c) \qquad \ldots\ x \cdot y^2 < 0\ \ldots$$

$\mathbb{B}$-propagate	-	$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*	$\mathbb{B}$-decision	$x \cdot y^2 < 0$
$\mathbb{B}$-propagate	-	$\mathbb{T}$-propagate	$x \in (-\infty, \infty)$
$\mathbb{B}$-decision	b = *false*	$\mathbb{T}$-decision	$x = 1$
$\mathbb{B}$-propagate	c = *true* ↯		
$\mathbb{B}$-conflict resolution	$(a \vee b)$		

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$ $\quad\quad \ldots\ x \cdot y^2 < 0\ \ldots$

$\mathbb{B}$-propagate	-	$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	a = *false*	$\mathbb{B}$-decision	$x \cdot y^2 < 0$
$\mathbb{B}$-propagate	-	$\mathbb{T}$-propagate	$x \in (-\infty, \infty)$
$\mathbb{B}$-decision	b = *false*	$\mathbb{T}$-decision	$x = 1$
$\mathbb{B}$-propagate	c = *true* $\lightning$	$\mathbb{T}$-propagate	$y \in \emptyset$ $\lightning$
$\mathbb{B}$-conflict resolution	$(a \vee b)$		

The MCSAT idea [de Moura, Jovanović, VMCAI'13]

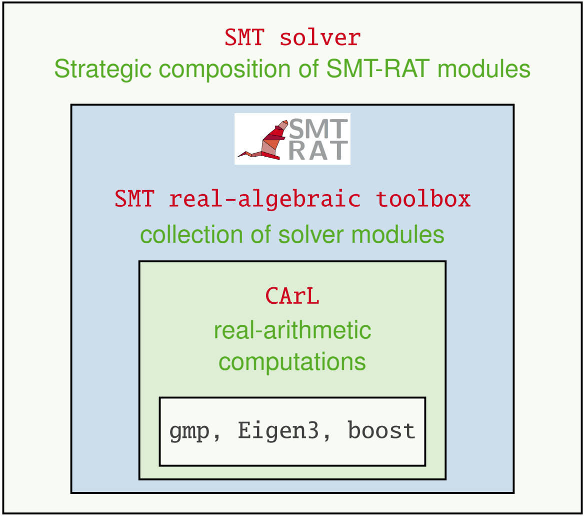
Exploration:	$\mathbb{B}$-decision	$\mathbb{T}$-decision
Look-ahead:	$\mathbb{B}$-propagation	$\mathbb{T}$-propagation
Proof system:	$\mathbb{B}$-conflict resolution	$\mathbb{T}$-conflict resolution

$(a \vee b \vee c) \wedge (a \vee b \vee \neg c)$ $\qquad$ $\ldots\ x \cdot y^2 < 0\ \ldots$

$\mathbb{B}$-propagate	-	$\mathbb{B}$-propagate	-
$\mathbb{B}$-decision	$a = false$	$\mathbb{B}$-decision	$x \cdot y^2 < 0$
$\mathbb{B}$-propagate	-	$\mathbb{T}$-propagate	$x \in (-\infty, \infty)$
$\mathbb{B}$-decision	$b = false$	$\mathbb{T}$-decision	$x = 1$
$\mathbb{B}$-propagate	$c = true$ ⚡	$\mathbb{T}$-propagate	$y \in \emptyset$ ⚡
$\mathbb{B}$-conflict resolution	$(a \vee b)$	$\mathbb{T}$-conflict resolution	$(x \cdot y^2 < 0 \rightarrow x < 0)$

Contents

Our SMT-RAT library [SAT'12, SAT'15]



- MIT licensed source code: `github.com/smtrat/smtrat`
- Documentation: `smtrat.github.io`

Solver modules in SMT-RAT [SAT'12, SAT'15]

CArL library: basic arithmetic datatypes and computations [Sapientia'18, NFM'11, CAI'11]

Basic modules

- SAT solver
- CNF converter
- Preprocessing/simplifying modules

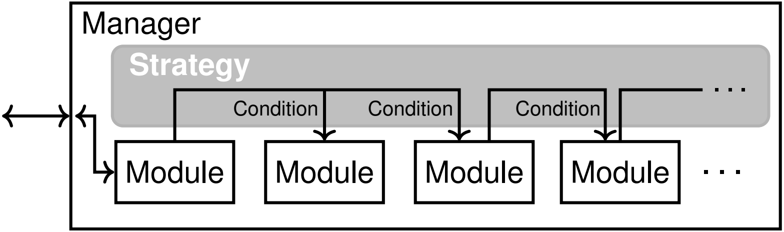
Non-algebraic decision procedures

- Equalities and uninterpreted functions
- Bit-vectors
- Bit-blasting
- Interval constraint propagation
- Pseudo-Boolean formulas

Algebraic decision procedures

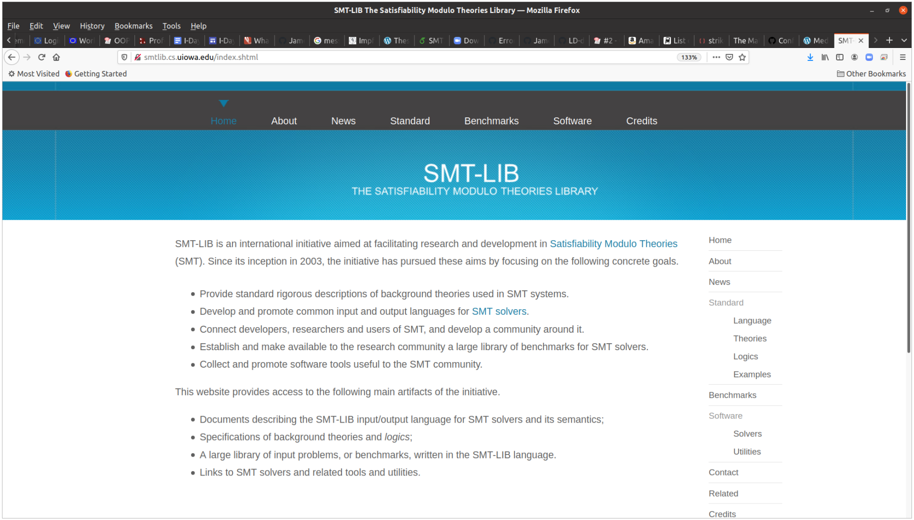
- Gauß+Fourier-Motzkin, FMplex [GandALF'23]
- Gröbner bases [CAI'13]
- MCSAT (FM,VS,CAD) [2xSC^2'19]
- Simplex [ISSAC'21]
- Cylindrical algebraic decomposition [SC^2'21, CADE-24, JSC'19, SC^2'17, 3 PhDs]
- Cylindrical algebraic covering [SMT'23, JLAMP'21, SYNASC'21, PhD Kremer]
- Virtual substitution [FCT'11, SC^2'17, 1 PhD]
- Subtropical satisfiability [NFM'23]
- Generalized branch-and-bound [CASC'16]
- Cube tests
- Linearization

Strategic composition of solver modules in SMT-RAT

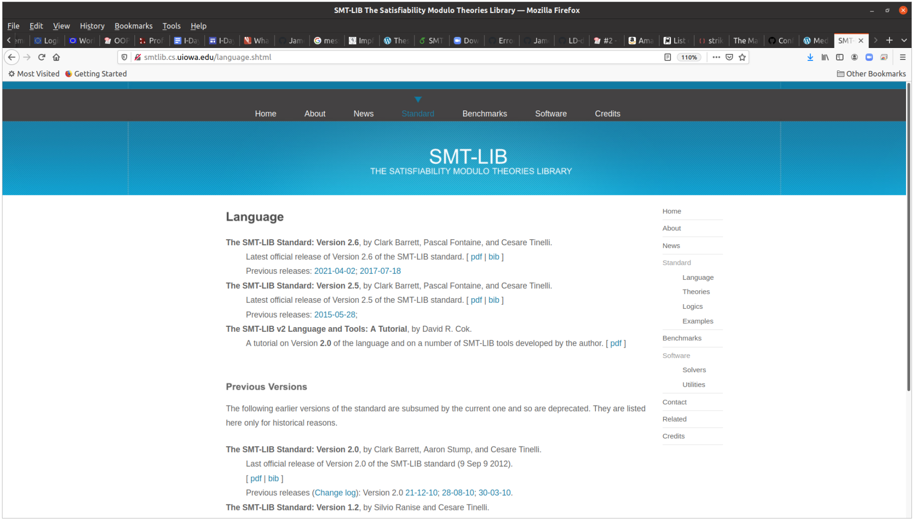


Contents

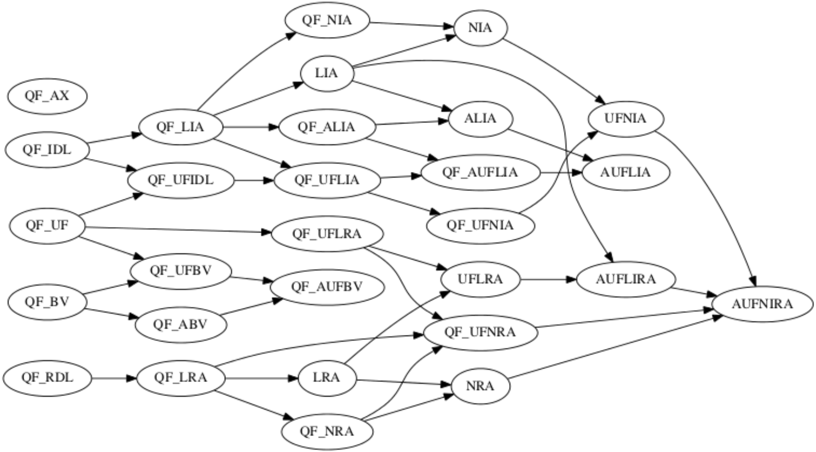
The Satisfiability Modulo Theories Library



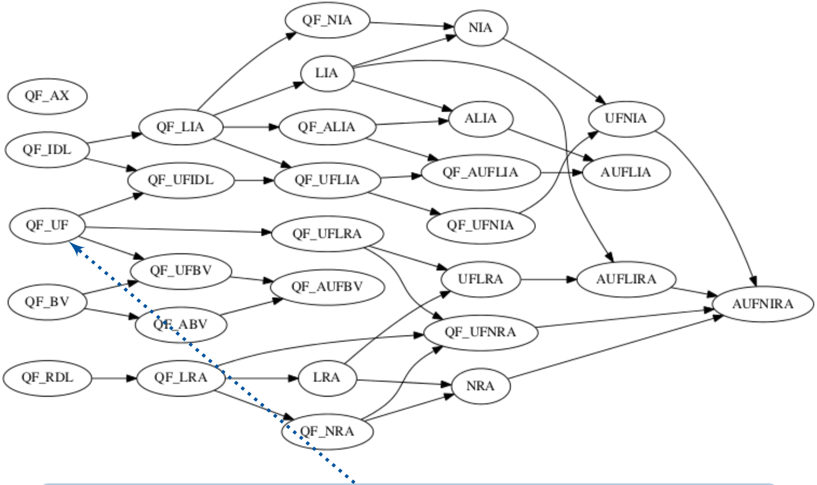
The Satisfiability Modulo Theories Library



SMT-LIB theories



Source: `http://smtlib.cs.uiowa.edu/logics.shtml`

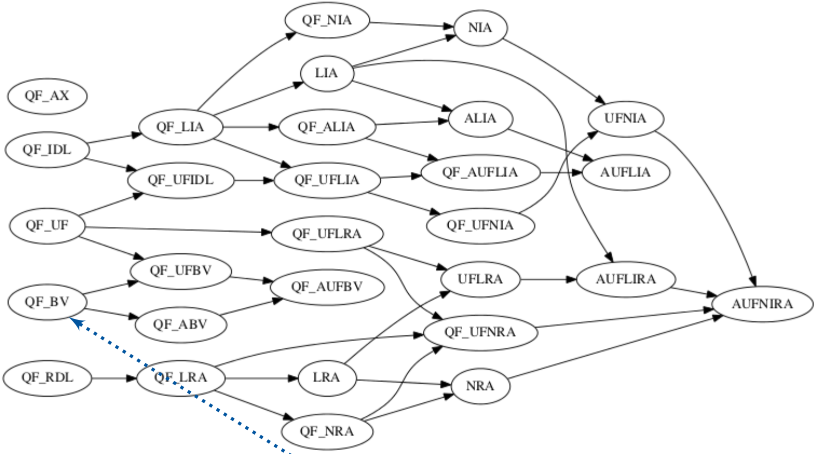


Quantifier-free equality logic with uninterpreted functions

$$(a = c \wedge b = d) \rightarrow f(a, b) = f(c, d)$$

Source: `http://smtlib.cs.uiowa.edu/logics.shtml`

SMT-LIB theories

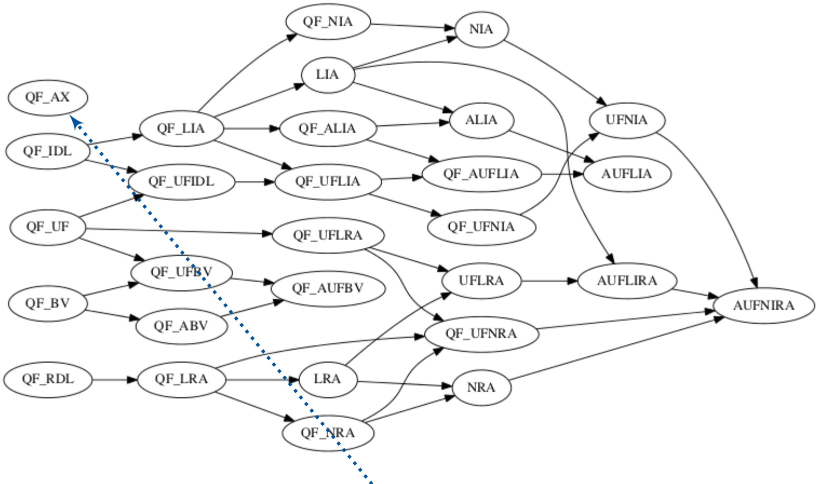


Quantifier-free bit-vector arithmetic

$(a|b) \leq (a\&b)$

Source: `http://smtlib.cs.uiowa.edu/logics.shtml`

SMT-LIB theories

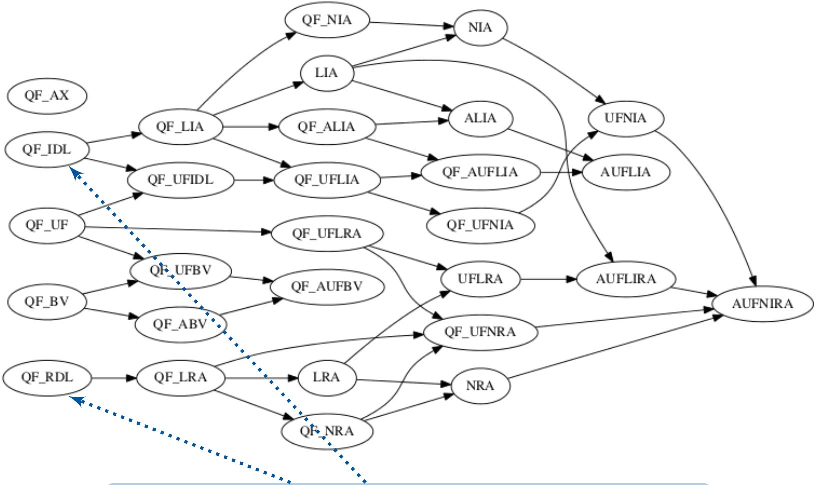


Quantifier-free array theory

$i = j \rightarrow \mathit{read}(\mathit{write}(a, i, v), j) = v$

Source: `http://smtlib.cs.uiowa.edu/logics.shtml`

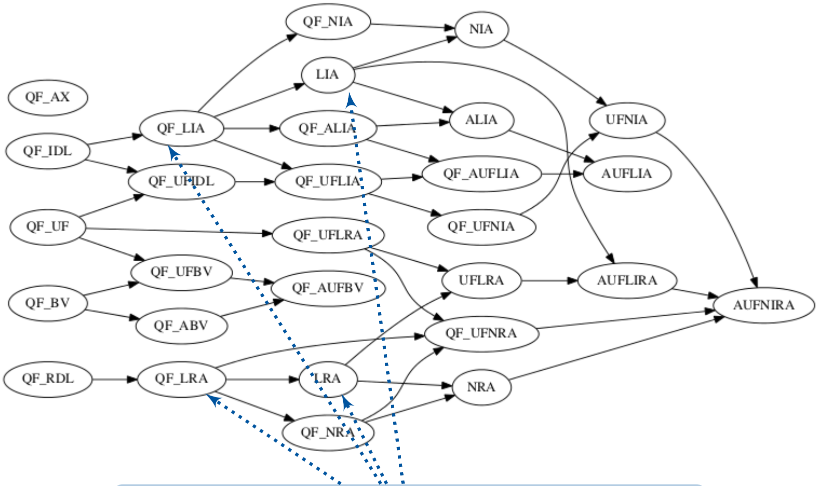
SMT-LIB theories



Quantifier-free integer/rational difference logic

$x - y \sim 0, \sim \in \{<, \leq, =, \geq, >\}$

Source: `http://smtlib.cs.uiowa.edu/logics.shtml`

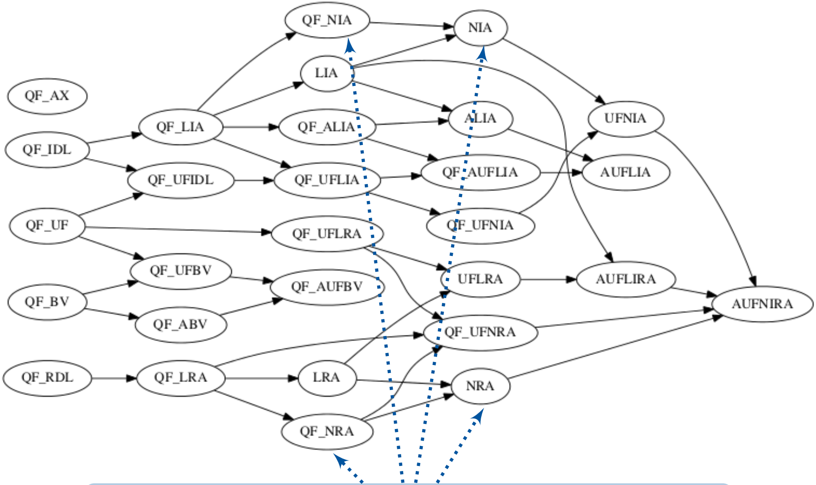


(Quantifier-free) real/integer linear arithmetic

$3x + 7y = 8$

Source: `http://smtlib.cs.uiowa.edu/logics.shtml`

SMT-LIB theories

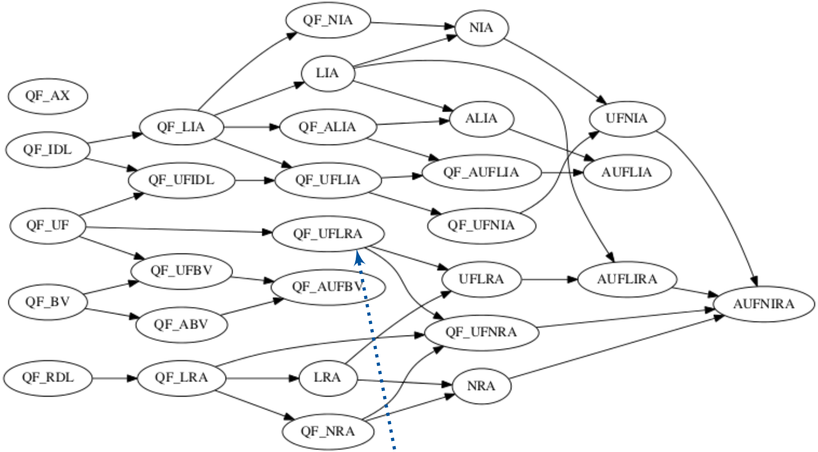


(Quantifier-free) real/integer non-linear arithmetic

$$x^2 + 2xy + y^2 \geq 0$$

Source: http://smtlib.cs.uiowa.edu/logics.shtml

SMT-LIB theories

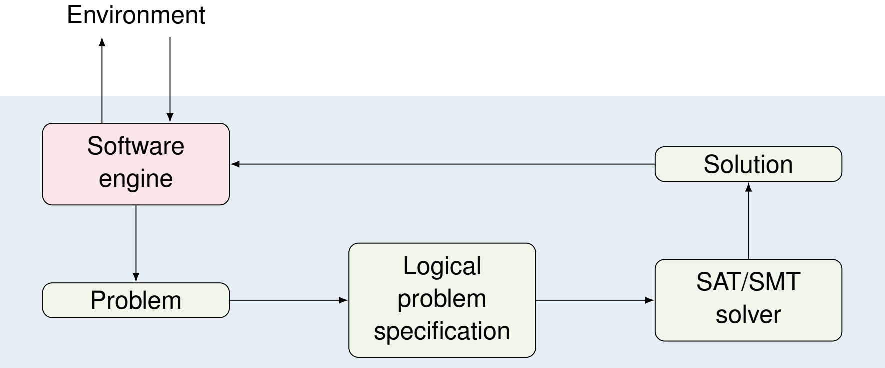


Combined theories
$2f(x) + 5y > 0$

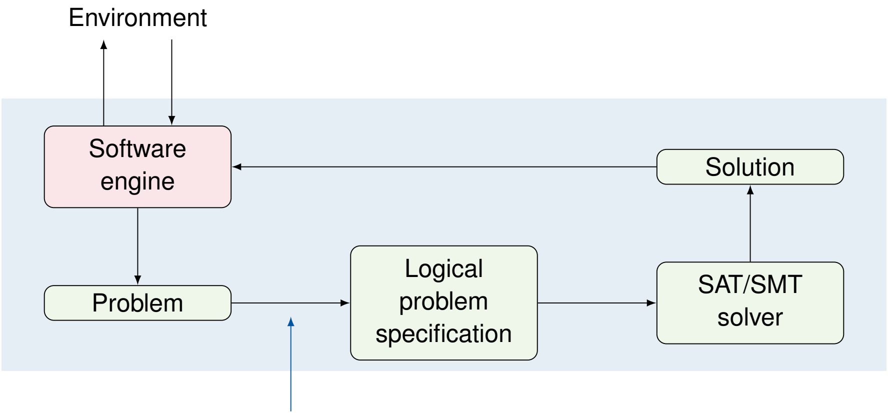
Source: `http://smtlib.cs.uiowa.edu/logics.shtml`

Contents

Embedding SAT/SMT solvers

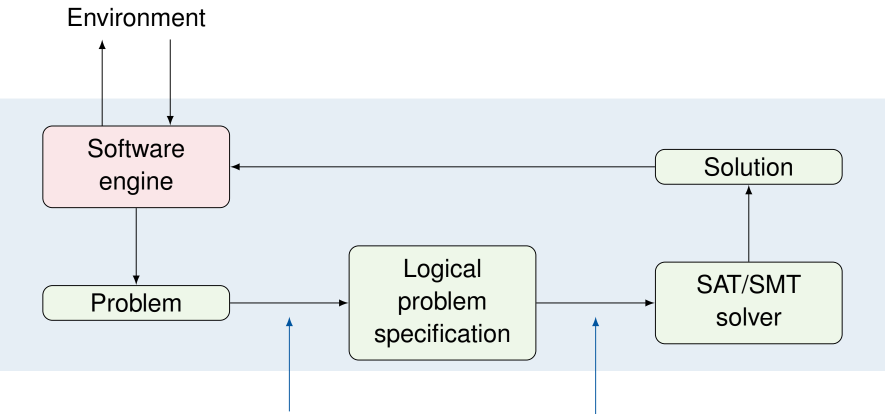


Embedding SAT/SMT solvers



Encoding: SAT/SMT-LIB standard
elaborate encoding is extremely important!

Embedding SAT/SMT solvers



Encoding: SAT/SMT-LIB standard
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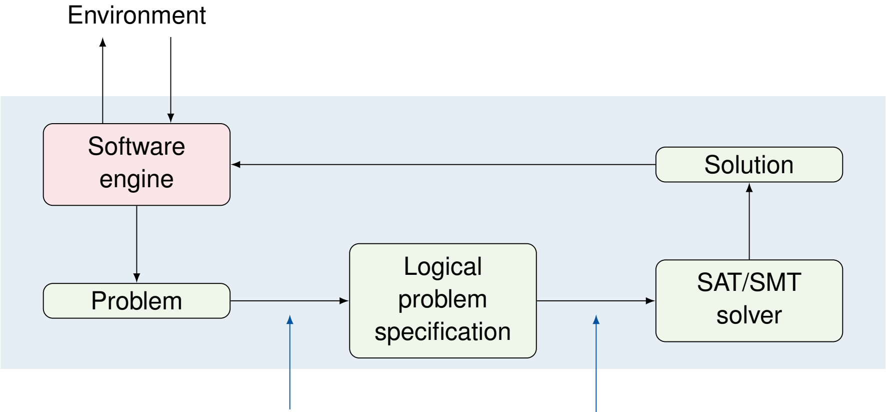
standard input syntax → free solver choice

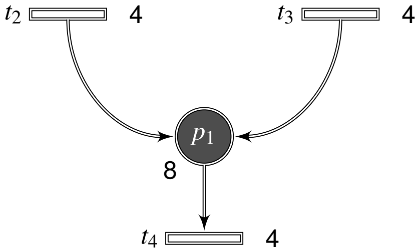
Embedding SAT/SMT solvers



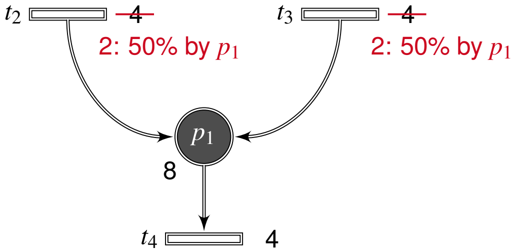
Encoding: SAT/SMT-LIB standard
elaborate encoding is extremely important!

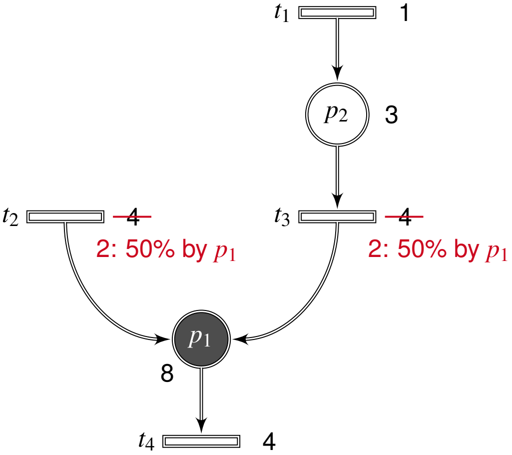
standard input syntax → free solver choice

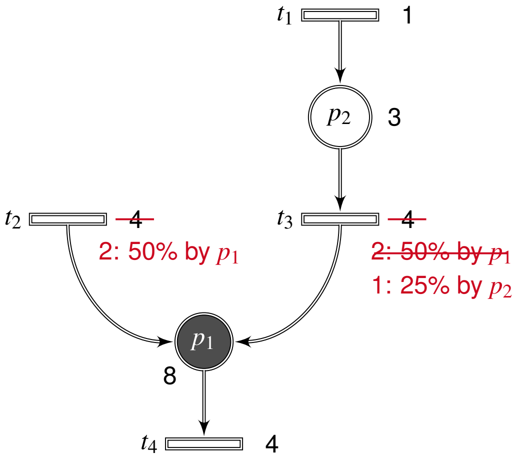
Next: some applications of SMT solvers


t_2
4
t_3
4
p_1
8
t_4
4

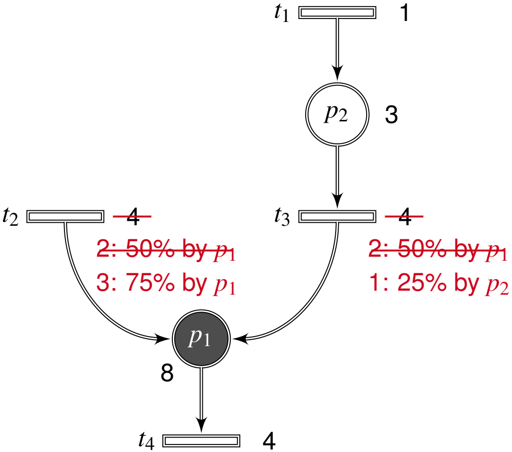
1. Rate adaption in hybrid Petri nets




t_1
1
p_2
3
t_2
~~4~~
2: 50% by p_1
t_3
~~4~~
2: 50% by p_1
p_1
8
t_4
4


t_1
1
p_2
3
t_2
~~4~~
2: 50% by p_1
t_3
~~4~~
~~2: 50% by p_1~~
1: 25% by p_2
p_1
8
t_4
4

1. Rate adaption in hybrid Petri nets



1. SMT encoding of rate adaption fixedpoint

$$(1) \quad \Big[\bigwedge_{p\in P} 0 \le \mathtt{factor}_p \le 1\Big] \wedge \Big[\bigwedge_{t\in T_a} 0 \le \mathtt{factor}_t \le 1\Big] \wedge$$

$$(2) \quad \Big[\bigwedge_{t\in T_a} ((\mathtt{owner}_t = \mathit{source}(t) \wedge \mathtt{owner}_t \in P_{\mathit{empty}}) \vee (\mathtt{owner}_t = \mathit{target}(t) \wedge \mathtt{owner}_t \in P_{\mathit{full}}))\Big] \wedge$$

$$(3) \quad \Big[\bigwedge_{p\in P} \mathtt{in}_p = (\textstyle\sum_{t\in \mathit{In}(p)\cap T_a} \mathtt{factor}_t \cdot \mathit{nominal\_rate}(t)) + (\sum_{t\in \mathit{In}(p)\cap T_{na}} \mathit{nominal\_rate}(t)) \wedge$$

$$\mathtt{out}_p = (\textstyle\sum_{t\in \mathit{Out}(p)\cap T_a} \mathtt{factor}_t \cdot \mathit{nominal\_rate}(t)) + (\sum_{t\in \mathit{Out}(p)\cap T_{na}} \mathit{nominal\_rate}(t))\Big] \wedge$$

$$(4) \quad \Big[\bigwedge_{p\in P_{\mathit{empty}}} \Big((\mathtt{factor}_p = 1 \vee \bigvee_{t\in \mathit{Out}(p)} \mathtt{owner}_t = p) \wedge$$

$$(\bigwedge_{t\in \mathit{Out}(p)} (\mathtt{owner}_t = p \rightarrow \mathtt{factor}_t = \mathtt{factor}_p) \wedge$$

$$(\mathtt{owner}_t \ne p \rightarrow \mathtt{factor}_t < \mathtt{factor}_p) \quad) \wedge$$

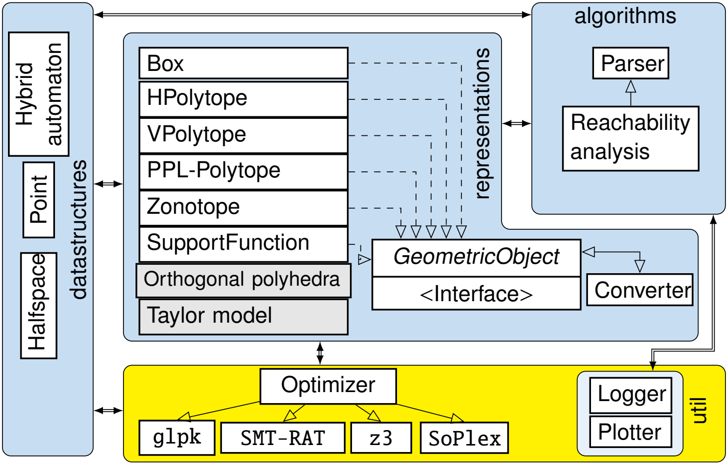
$$\mathtt{in}_p \ge \mathtt{out}_p \wedge (\mathtt{factor}_p < 1 \rightarrow \mathtt{in}_p = \mathtt{out}_p) \quad \Big)\Big] \wedge$$

$$(5) \quad \Big[\bigwedge_{p\in P_{\mathit{full}}} \Big((\mathtt{factor}_p = 1 \vee \bigvee_{t\in \mathit{In}(p)} \mathtt{owner}_t = p) \wedge$$

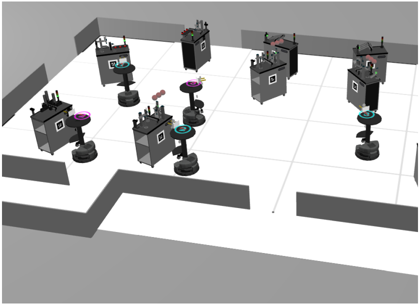
$$(\bigwedge_{t\in \mathit{In}(p)} (\mathtt{owner}_t = p \rightarrow \mathtt{factor}_t = \mathtt{factor}_p) \wedge$$

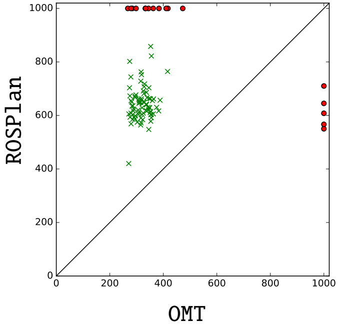
$$(\mathtt{owner}_t \ne p \rightarrow \mathtt{factor}_t \le \mathtt{factor}_p) \quad) \wedge$$

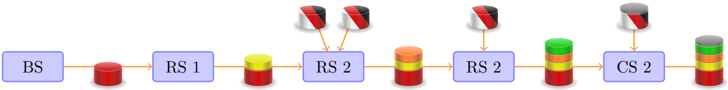
$$\mathtt{in}_p \le \mathtt{out}_p \wedge (\mathtt{factor}_p < 1 \rightarrow \mathtt{in}_p = \mathtt{out}_p) \quad \Big)\Big]$$



Source: E. Ábrahám, X. Chen, S. Sankaranarayanan, S. Schupp. PhD Chen, PhD Schupp, Information and Computation'22, IRI'18, SEFM'18, TACAS'18, NFM'17, QAPL'17, ARCH'15, CyPhy'15, NFM'15, FMCAD'14, CAV'13, FTSCS'13, NOLCOS'13, RTSS'12, EUROCAST'11, RP'11.

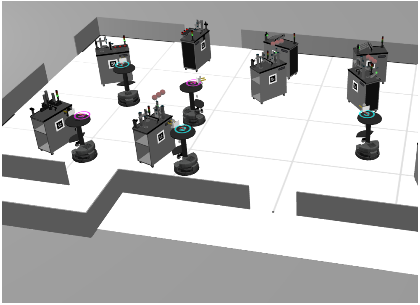


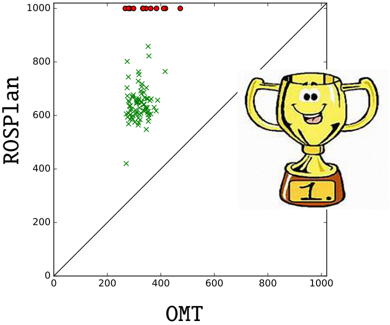


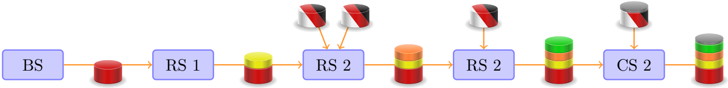


Source: E. Ábrahám, G. Lakemeyer, F. Leofante, T. D. Niemüller, A. Tacchella.

PhD Leofante, IJCAI'20, Information Systems Frontiers 2019, ECMS'19, AAAI'18, iFM'18, ICAPS'17, PlanRob'17, IRI'17.







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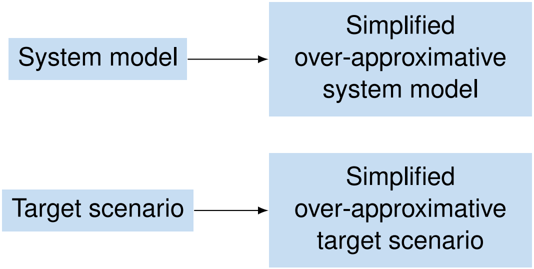
PhD Leofante, IJCAI'20, Information Systems Frontiers 2019, ECMS'19, AAAI'18, iFM'18, ICAPS'17, PlanRob'17, IRI'17.

4. Relevant domains for testing (Siemens)

System model

Target scenario

4. Relevant domains for testing (Siemens)



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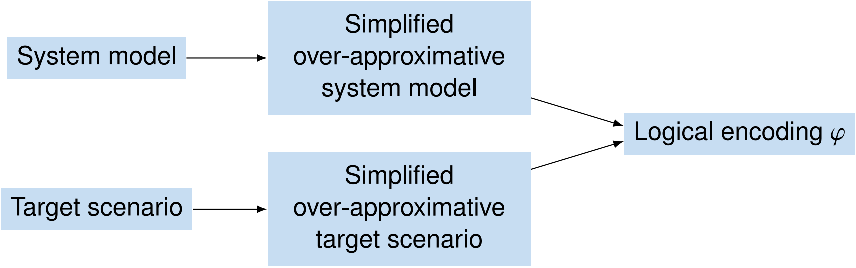

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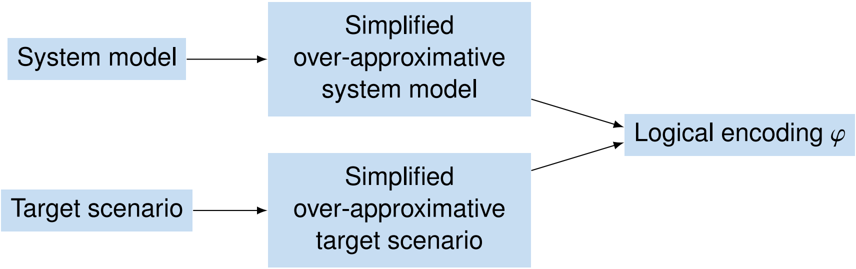
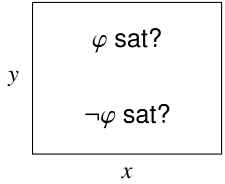
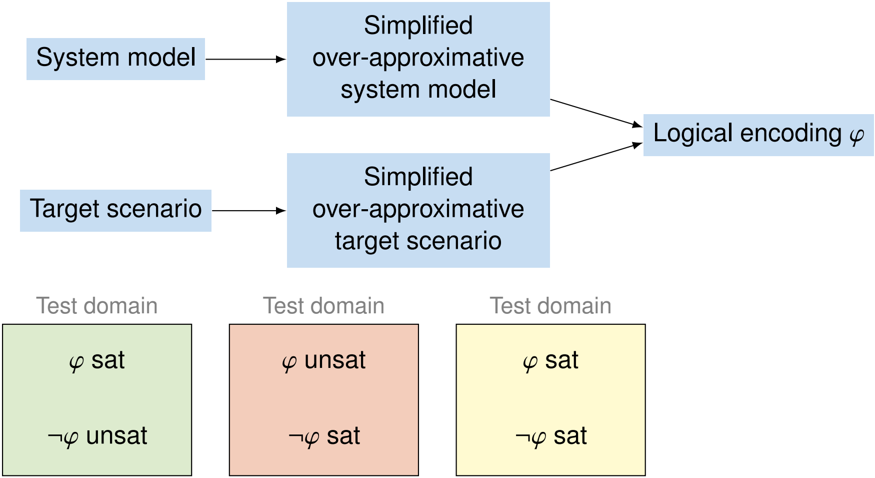
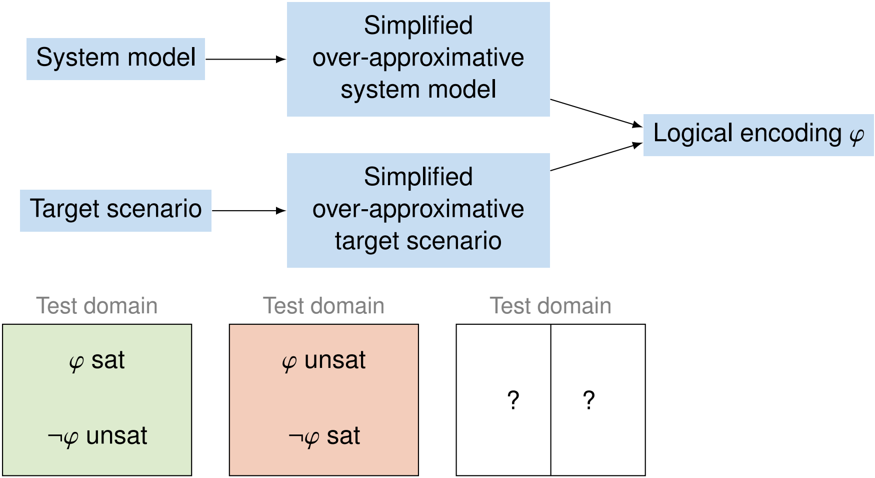


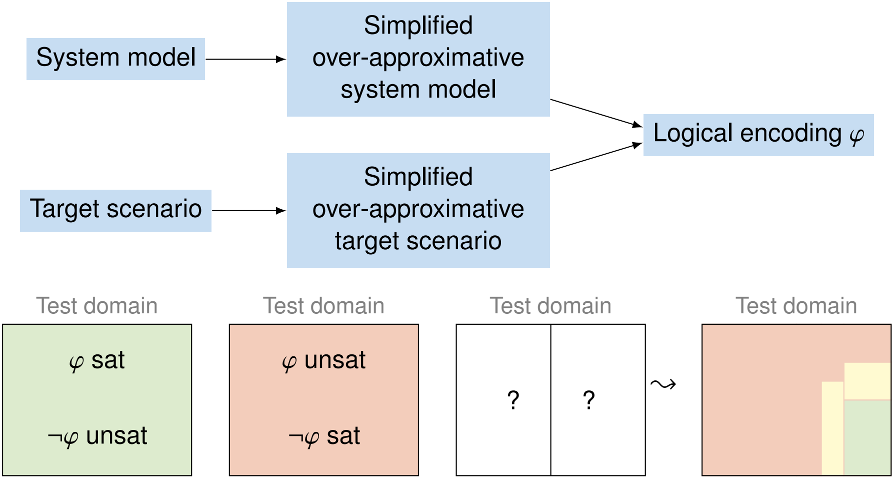
Test domain

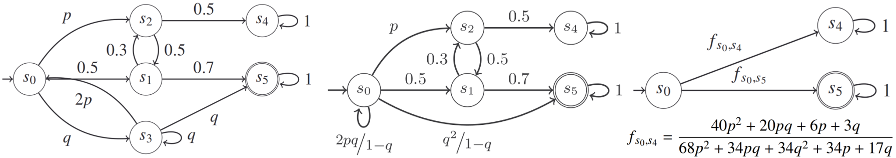


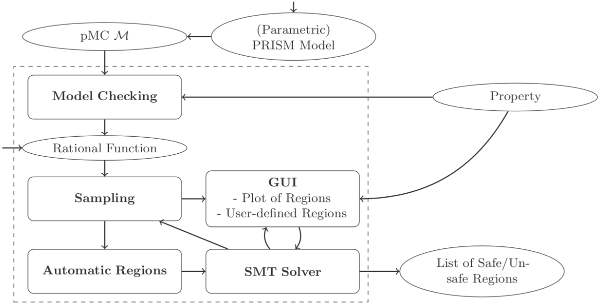

System model
Simplified
over-approximative
system model
Target scenario
Simplified
over-approximative
target scenario
Logical encoding φ
Test domain
φ sat
$\neg\varphi$ unsat
Test domain
φ unsat
$\neg\varphi$ sat
Test domain
φ sat
$\neg\varphi$ sat

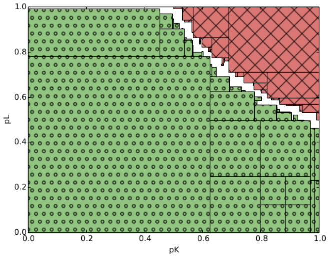
4. Relevant domains for testing (Siemens)




System model
Simplified over-approximative system model
Target scenario
Simplified over-approximative target scenario
Logical encoding φ
Test domain
φ sat
$\neg\varphi$ unsat
Test domain
φ unsat
$\neg\varphi$ sat
Test domain
?
?
$\rightsquigarrow$
Test domain







Source: C. Dehnert, S. Junges, N. Jansen, F. Corzilius, M. Volk, H. Bruintjes, J.-P. Katoen, E. Ábrahám.

PROPhESY: A probabilistic parameter synthesis tool.

In Proc. of CAV'15.

Contents

Usage of SMT solvers

- Standard input language, benchmarks
- Online usage, command-line, programming interfaces
- Black-box usage possible, but specific knowledge is advantageous
  - for efficient usage and
  - selection of the best fitting tool (e.g. fast vs complete).

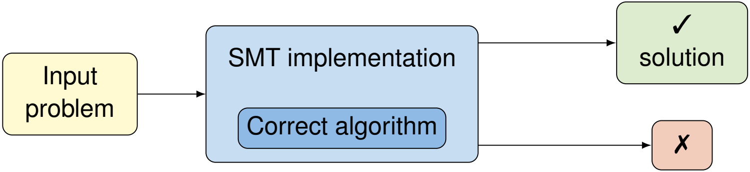
Proof generation

- Theoretical basics: algorithms with correctness proofs.

Correct algorithm

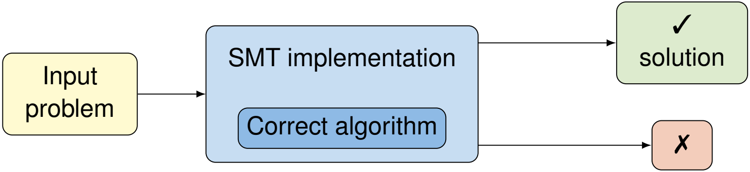
Proof generation

- Theoretical basics: algorithms with correctness proofs.
- Reliable tools: in QF_NRA for SMT-COMP'21, no bugs discovered on large benchmark sets.



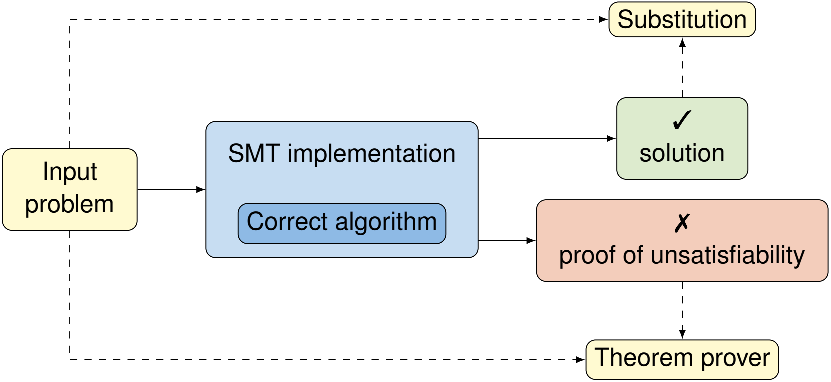
Proof generation

- Theoretical basics: algorithms with correctness proofs.
- Reliable tools: in QF_NRA for SMT-COMP'21, no bugs discovered on large benchmark sets.
- But still: bugs can remain undetected for a long time.



Proof generation

- Theoretical basics: algorithms with correctness proofs.
- Reliable tools: in QF_NRA for SMT-COMP'21, no bugs discovered on large benchmark sets.
- But still: bugs can remain undetected for a long time.
- Solution: automatically checkable proof certificates.



Further functionalities

- Model generation
- Explanations of unsatisfiability (unsat cores, interpolants)
- Optimization

- Satisfiability for quantified formulas
- Quantifier elimination (get all solutions symbolically)

- Scalability
  - Preprocessing
  - Heuristics, especially variable ordering
  - Machine learning
  - Closer integration of decision procedures
  - Parallelization

Contents

SMT-LIB theories

Syntax of core theory

```
      :sorts ((Bool 0))
      :funs (
(true Bool)
(false Bool)
(not Bool Bool)
(and Bool Bool Bool :left-assoc)
...
(par (A) (= A A Bool :chainable))
(par (A) (ite Bool A A A))
...
```

SMT-LIB theories

Syntax of real theory

```
       :sorts ((Real 0))
       :funs (
...
(+ Real Real Real :left-assoc)
(* Real Real Real :left-assoc)
...
(<  Real Real Bool :chainable)
...
       )
```

SMT-LIB commands

- Lisp-like script language
- Supported by essentially all SMT solvers
- Easy to parse and extend

Boolean example

```
(set-logic QF_UF)
(declare-const p Bool)
(assert (and p (not p)))
(check-sat)
```

SMT-LIB commands

- Lisp-like script language
- Supported by essentially all SMT solvers
- Easy to parse and extend

Linear integer example

```
(set-logic QF_LIA)
(declare-const x Int)
(declare-const y Int)
(assert (= (- x y) (+ x (- y) 1)))
(check-sat)
```

SMT-LIB commands

- Lisp-like script language
- Supported by essentially all SMT solvers
- Easy to parse and extend

Unsatisfiable cores

```
(set-logic QF_UF)
(set-option :produce-unsat-cores true)
(declare-const p Bool)
(declare-const q Bool)
(declare-const r Bool)
(assert (! (=> p q) :named a))
(assert (! (=> q r) :named b))
(assert (! (not (=> p r)) :named c))
(assert ...)
(check-sat)
(get-unsat-core)
```

SMT-LIB commands

- Lisp-like script language
- Supported by essentially all SMT solvers
- Easy to parse and extend

Optimization

```
(set-logic QF_LIA)
(declare-const x Int)
(declare-const y Int)
(assert (and (< y 5) (< x 2)))
(assert (< (- y x) 1))
(maximize (+ x y))
(check-sat)
(get-objectives)
```

Solving theory formulas with SMT solvers

- https://cvc5.github.io/tutorials/beginners
- SMT-LIB input:
  https://microsoft.github.io/z3guide/docs/logic/intro/
  https://smt-lib.org/examples.shtml
- Z3/cvc5 Python interface:
  https://ericpony.github.io/z3py-tutorial/guide-examples.htm