Developing and Proving Algorithms with PVS
(Part II)

César A. Muñoz

NASA's Langley Research Center
Cesar.A.Munoz@nasa.gov
http://shemesh.larc.nasa.gov/people/cam

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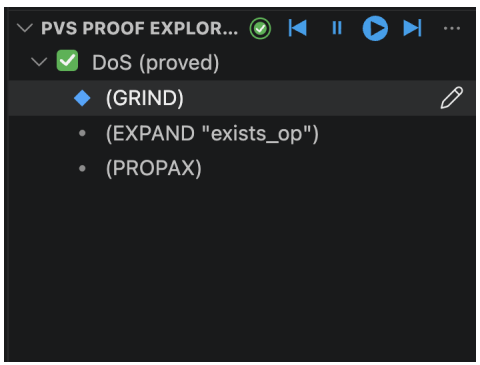
Recap: UAS Service Supplier

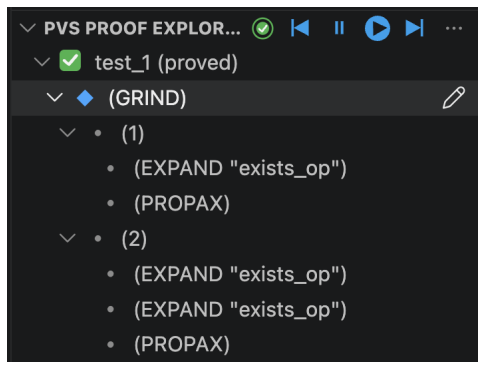
USS provides surveillance services for a set of aircraft within a given operational volume.

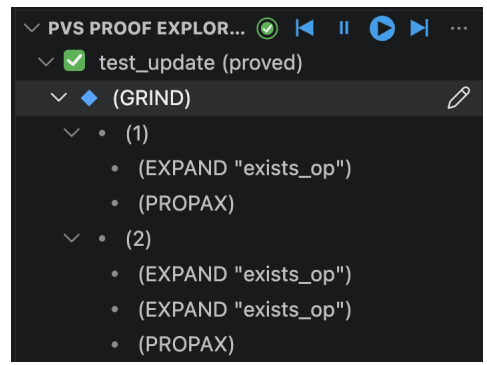
- Create USS, including maximum capacity and operational volume (area & height).
- Subscribe/unsubscribe aircraft operations.
- Check if an aircraft is subscribed.
- Get information about aircraft operation.
- Check if maximum capacity has been reached.
- Update aircraft state information (position and velocity).
- Retrieve aircraft state.
- Monitor traffic conflicts.
- Monitor keep-in geofencing.

Previously on This Talk . . .

- Types, including records, sub-types, and dependent types.
- Function, including recursive functions.
- Inductive predicates.
- Type judgements, including recursive type judgements.
- Proofs:







(On Proofs, Proof Rules, and Strategies . . .

- Proofs are built in a procedural style using proof commands, which interactively construct a tree of sequents of the form $\Gamma \vdash \Delta$, where Γ and Δ are lists of formulas.
- In a sequent $\Gamma \vdash \Delta$, the list of formulas in Γ is called the *antecendent* and the list of formula in Δ is called the *consequent*.
- The antecent represents a conjunction and the consequent represents a disjunction, i.e., if $\Gamma = A_1, \ldots, A_n$ and $\Delta = B_1, \ldots B_m$, $\Gamma \vdash \Delta$ represents the formula

$$\bigwedge_{0 \leq i \leq n} A_i \implies \bigvee_{0 \leq i \leq m} B_i$$

Proof Construction

- The root of the tree is the initial goal.
- Proof rules, which are applied bottom up, transform a sequent $\Gamma \vdash \Delta$ into a list of sub-goal sequents $\Gamma_i \vdash \Delta_i$, i.e.,

$$\frac{\Gamma_1 \vdash \Delta_1 \quad \ldots \quad \Gamma_k \vdash \Delta_k}{\Gamma \vdash \Delta} \text{ (proof-rule)}$$

- The construction ends when all the leaves are discharged, i.e., they have the form $\ldots \vdash \ldots, \mathit{TRUE}, \ldots$.
- Remark: Rules are not bi-directional, arriving to an empty sequent "$\vdash$ doesn't mean the goal is false.
- Some proof rules may not terminate.

Proof Rules[1]

- `(assert ...)` : Apply decision procedures and auto-rewrites.
- `(grind ...)`: Super-duper strategy.
- `(ground ...)`: Apply propositional simplifications.
- `(flatten ...)` : Eliminate conjunctions in the antecedent and disjunctions in the consequent
- `(skeep ...)`: Eliminate `FORALL` quantifier in the consequent or `EXISTS` quantifier in the antecedent by introducing skolem variables.
- `(lemma ...)`: Introduce lemma to the consequent.
- `(expand ...)`: Expand definition.
- `(inst ...)`: Eliminate `FORALL` quantifier in the antecendent or `EXISTS` quantifier in consequent.
- ...

[1] See `https://pvs.csl.sri.com/doc/pvs-prover-guide.pdf` for a more comprehensive list of proof commands.

Proof Commands

- Proof Rule: Atomic (trusted) prover command, e.g., `split`, `skolem`, `hide`, `metit`, etc.
- Strategy: A proof command that expands into one or more atomic steps, e.g., `grind`, `ground`, etc.
  - Black Box: Proof command that behaves as an atomic step but can be expanded, e.g., `grind`, `ground`, `interval`, etc.
  - Glass Box: A command that is always expanded, e.g., strategy combinators such as `then`, `if`, `try`,...[2]
  - Tactic: A strategy defined inside a proof.

[2] Black box strategies have a "$"-glass box variant, e.g., `grind$`, `ground$`, `interval$`, ...

User-Defined Strategies

- User-defined *strategies* conservatively extend the theorem prover capabilities.
- Strategies are defined in a macro-based language implemented in Lisp.[3]
- Strategies do not compromise the soundness of PVS (as long as the proof context is accessed as read-only).

[3] Lisp is the implementation language of PVS.

Strategy Language: Basic Steps

- Any proof command, e.g., `(ground)`, `(case ...)`, etc.
- `(skip)` does nothing.
- `(printf format ...)` prints a formatted `message`.
- `(comment message)` adds a persistent comment to the proof branch.
- `(relabel label fnums)` labels formulas `fnums` with `label`.
- `(delabel fnums)` unlabels formulas in `fnums`.
- . . . and much more!

Strategy Language: Combinators

- Sequencing: `(then step1 ...stepn)`
- Branching: `(branch step (step1 ...stepn))`
- Binding local variables:
  `(let ((var1 lisp1) ...(varn lispn)) step)`
- Conditional: `(if lisp step1 step2)`
- Loop: `(repeat step)`
- Backtracking: `(try step step1 step2)`

Remark: `if` and `let` are the only combinators where Lisp code can be used.

Creating Fresh Labels

(Advanced Topic)

- `(with-fresh-labels ((var1 fnum1) ...(varn fnumn)) steps)`:
  Create fresh labels for `fnumi` and binds them to `vari`. Then, sequentially apply `steps` to all branches. All created labels are removed before the strategy exits.
- Example:

```
(with-fresh-labels
  ((l 1) (m -1))
  (inst? l :where m))
```

Creating Fresh Names

(Advanced Topic)

- `(with-fresh-names ((var1 e1) ...(varn en)) steps)`: Create fresh names for `ei` and binds them to `vari`. Then, sequentially apply `steps` to all branches. All created names are removed before the strategy exits.
- Example:

```
(with-fresh-names
  ((e "x+2") (f "sqrt(x)"))
  (inst 1 e f))
```

Writing your Own Strategies

- New strategies are defined in a file named `pvs-strategies` in the current context.
- PVS automatically loads this file every time the theorem prover is invoked.
- The `IMPORTING` clause automatically loads any file `pvs-strategies` in the importing chain.

Strategy Definitions

- `defstep` defines a black-box strategy and its glass-box $-form:

```
(defstep name (parameters &optional parameters)
  step
  help-string  format-string)
```

- `defhelper` defines a black-box strategy to be used by other strategies.

```
(defhelper name (parameters &optional parameters)
  step
  help-string  format-string)
```

- `defstrat` defines a glass-box strategy:

```
(defstrat name (parameters &optional parameters)
  step
  help-string)
```

Example: try-grind

In pvs-strategies:

```
(defstep try-grind ()
  (else (then (grind)(fail)) (skip))
  "Tries GRIND. UNDO, if it doesn't discharge
   the sequent"
  "Trying GRIND")
```

Example: try-grind

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```

```
  |-------
{1}   x * (1 + x) >= 0

Rule? (help try-grind)
(try-grind/$) :
Tries GRIND. UNDO, if it doesn't discharge the sequent

Rule? (try-grind)
No change on: (try-grind)
test :

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```
No change on: (try-grind)
```
test :

  |-------
{1}   x * (1 + x) >= 0
```

```
test.1 :

{-1}  x >= 0
  |-------
[1]   x * (1 + x) >= 0

Rule? (try-grind)
Trying GRIND,

This completes the proof of test.1.
```

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test.1 :

{-1}  x >= 0
  |-------
[1]   x * (1 + x) >= 0

Rule? (try-grind)
```
Trying GRIND,

```
This completes the proof of test.1.
```

Glass-Box vs Black-Box Strategies

- `try-grind` was defined as black-box strategy. Therefore, it is saved in the proof even when it skips.
- When `try-grind` skips, it would be better not to save the command `grind`, which is expensive.
- Two alternatives:
  - Use `try-grind$` instead of `try-grind`.
  - Define `try-grind` as a glass-box strategy:

```
(defstrat try-grind ()
  (else (then (grind)(fail)) (skip))
  "Tries GRIND. UNDO, if it doesn't discharge
   the sequent")
```

Level of Strategy Development

Easy Repetitive tasks, e.g., `(flatten)(assert)(replace -1)...(flatten)(assert)(replace -1)`.

Medium Programatic tasks, e.g., `(case "0 <= n AND n < 1")`, ..., `(case "1023 <= n AND n < 1024")`.

Advanced Control flow, e.g., implementation of a new proof combinator.

Expert Proof search, e.g., implementation of a decision procedure or an heuristic method.

Using Lisp to Access PVS Proof Context

- Arbitrary Lisp expressions (functions, global variables, etc.) can be included in a strategy file.
- PVS's data structures are based on various Common Lisp Object System (CLOS) classes. They are available to the strategy programmer through global variables and accessory functions.

Strategy Development Pitfalls

- Strategies may not terminate, e.g.,

  `Rule? (repeat (case "0=0"))`
- Strategies may be non-deterministic, e.g.,

  `Rule? (let ((n (freshname "n"))) (name n "10"))`
- Avoid non-deterministic black-box strategies; they may fail when the proof is rerun.
- If non-determinism is unavoidable, use glass-box strategies.
- If fresh identifiers are needed, use the robust glass-box strategies `with-fresh-labels` and `with-fresh-names`.

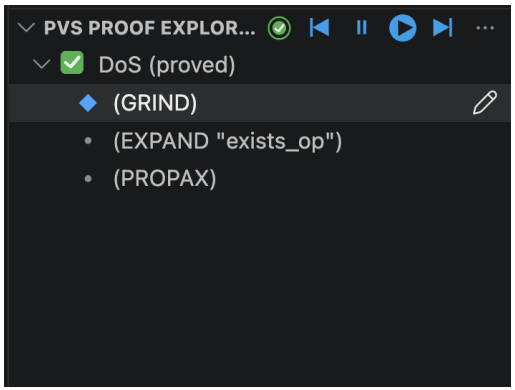
. . . References)

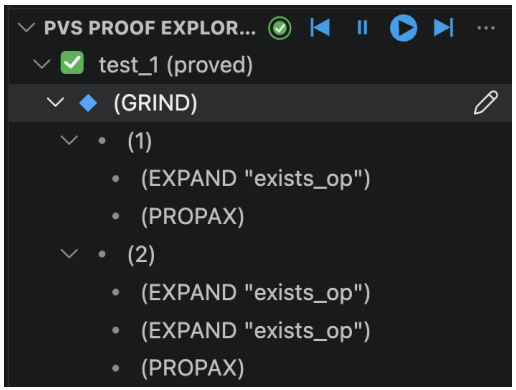
- Documentation: PVS Prover Guide, N. Shankar, S. Owre, J. Rushby, D. Stringer-Calvert, SRI International: `http://www.csl.sri.com/pvs.html`.
- Proceedings of STRATA 2003: `http://hdl.handle.net/2060/20030067561`.
- Examples: Manip[4], Field[5], Extrategies[6].
- Programming: Lisp The Language, G. L. Steele Jr., Digital Press[7].

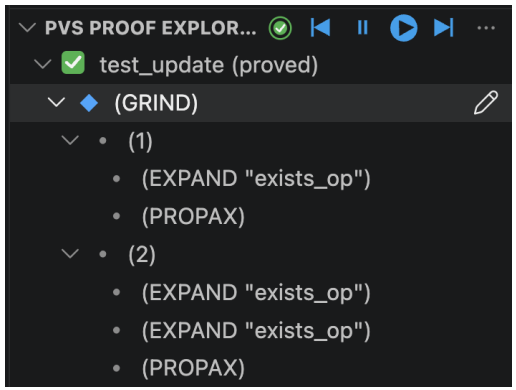
[4] `https://shemesh.larc.nasa.gov/fm/pvs/Manip/`
[5] `https://shemesh.larc.nasa.gov/fm/pvs/Field/`
[6] `https://shemesh.larc.nasa.gov/fm/pvs/Extrategies/`
[7] `https://www.cs.cmu.edu/Groups/AI/html/cltl/clm/clm.html`

Exercise 6

Write a strategy called `uss-tcc` that discharges the proofs below and use it to prove `subscribe_j`, `test_0`, `test_1`, DoS, and `test_update`.







Hint: The strategy does the following: first apply `grind`, then to each generated subgoal, expand `exists_op` as many times as needed.

Is This Model Still Useful?

```
UTMO[AircraftOp:TYPE+,
     OperationalArea:TYPE+] : THEORY
BEGIN

  % Type alias
  AircraftId : TYPE = string

  % Association list of tuples (id,info)
  Operations : TYPE = list[[AircraftId,AircraftOp]]

  % Record type
  USS : TYPE = [#
    max_capacity : posnat,
    op_alt  : nnreal,
    op_area : OperationalArea,
    operations : Operations
  #]
```

What About This One?

```
UTM1[AircraftOp:TYPE+,
     OperationalArea:TYPE+] : THEORY
BEGIN

  % Type alias
  AircraftId : TYPE = string

  % Association list of tuples (id,info)
  Operations : TYPE = [AircraftId->AircraftOp]

  ...
```

Maybe This One?

(Pun Intended)

```
UTM2[AircraftOp:TYPE+,
     OperationalArea:TYPE+] : THEORY
BEGIN

  IMPORTING structures@Maybe

  % Type alias
  AircraftId : TYPE = string

  % Association list of tuples (id,info)
  Operations : TYPE = [AircraftId->Maybe[AircraftOp]]

  ...
```

Maybe This One?

(Pun Intended)

```
UTM2[AircraftOp:TYPE+,
     OperationalArea:TYPE+] : THEORY
BEGIN

  IMPORTING structures@Maybe

  % Type alias
  AircraftId : TYPE = string

  % Association list of tuples (id,info)
  Operations : TYPE = [AircraftId->Maybe[AircraftOp]]

  ...
```

Or This One

```
UTMn[AircraftOp:TYPE+,
     OperationalArea:TYPE+] : THEORY
BEGIN

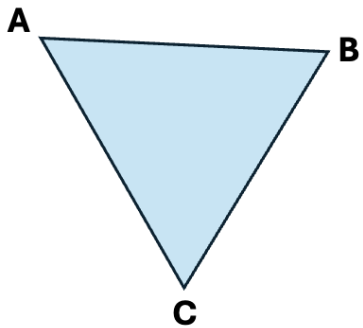
  IMPORTING HashMap

  AircraftId : TYPE = string

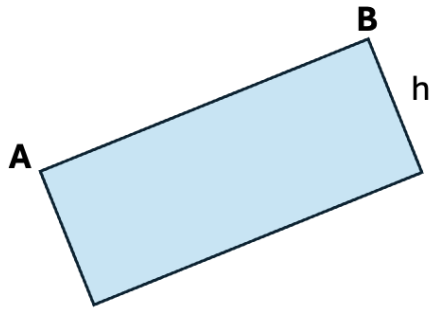
  Operations : TYPE = HashMap[AircraftId,AircraftOp]

 ...
```

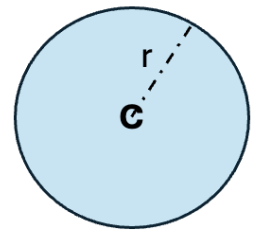
Operational Area




```
mk_triangle(A,B,C)
: shapes@Triangle_2D
```




```
mk_rectangle(A,B,h)
: shapes@Rectangle_2D
```




```
mk_circle(C,r)
: shapes@Circle_2D
```

Data Types

```
UTM : THEORY
BEGIN
  IMPORTING shapes@triangle_2D,
            shapes@circle_2D,
            shapes@rectangle_2D

  OperationalArea : DATATYPE
  BEGIN
    CircularArea(get_circle:Circle_2D)
        : circular_area?
    RectangularArea(get_rectangle
        : Rectangle_2D): rectangular_area?
    TriangularArea(get_triangle:Triangle_2D)
        : triangular_area?
  END OperationalArea
```

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UTM : THEORY
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  IMPORTING shapes@triangle_2D,
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        : Rectangle_2D): rectangular_area?
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        : triangular_area?
  END OperationalArea
```

Enumeration Types

```
UASClass : TYPE = { SMALL, MEDIUM, LARGE }
```

Enumeration types are syntactic sugar for a data type, i.e.,

```
UASClass : DATATYPE =
BEGIN
    SMALL: SMALL?
    MEDIUM: MEDIUM?
    LARGE: LARGE?
END UASClass
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```

(Inductive Data Types)

- Data types are inductively defined as the smallest fix point of objects built only using the constructor functions.
- Data types can be parametric.
- Data types generate several definitions, e.g., `ord`, `BinaryTree_induction`, `every`, `some`, `<<`, ... (see `BinaryTree_adt.pvs`).

```
%% File: BinaryTree.pvs
BinaryTree[T:TYPE] : DATATYPE
BEGIN
  Leaf(leaf:T) : leaf?
  Cons(left,right:BinaryTree) : cons?
END BinaryTree
```

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Putting Everything Together

```
IMPORTING vectors@vect_3D_2D
%% Vect3 : TYPE = [# x:real, y:real, z:real #]
FlightPlan : TYPE = list[Vect3]

AircraftOp : TYPE = [#
  uas_class   : UASClass,
  callsign    : string,
  operator_id : string,
  flight_plan : FlightPlan
#]

IMPORTING UTMO[AircraftOp,OperationalArea]
```

Exercise 7

Define the function `in_operational_volume?(uss:USS)(p:Vect3) : bool` that returns `TRUE` if the point `p` is inside the operational volume of `uss`.
Hints: Check that

1. The vertical component of `p`, i.e., `p`‘`z`, is below `uss`‘`op_alt`.
2. The horizontal component of `p`, i.e., `vect2(p)`, is inside `uss`‘`op_area`.

Look for the names of the functions in `NASALib/shapes` that check if a point is inside a circle, rectangle, and triangle.

Exercise 8

Define the function `check_flight_plan(uss:USS,ac_id:AircraftId)` that returns

- `NotFound`, if the aircraft `ac_id` is not in `uss`, or
- `OK`, if the aircraft is in `uss` and every waypoint in its flight plan lies inside the USS operational volume, or
- `Failure(wps)` otherwise, where `wps` is the list of waypoints in its flight plan that lie outside the USS operational volume.

Let's Animate

PVSio is a PVS utility that provides a read-eval loop for animation of functional specifications.

```
;; In UTM.pvs
USSC : USS = mk_uss(100, % Max capacity
                    150, % Operational altitude in m
                    LET C : Vect2 = (0,0) IN
                    CircularArea(mk_circle(C,2000)))

<PVSio> in_operational_volume(USSC)(0,0,0);
==>
TRUE

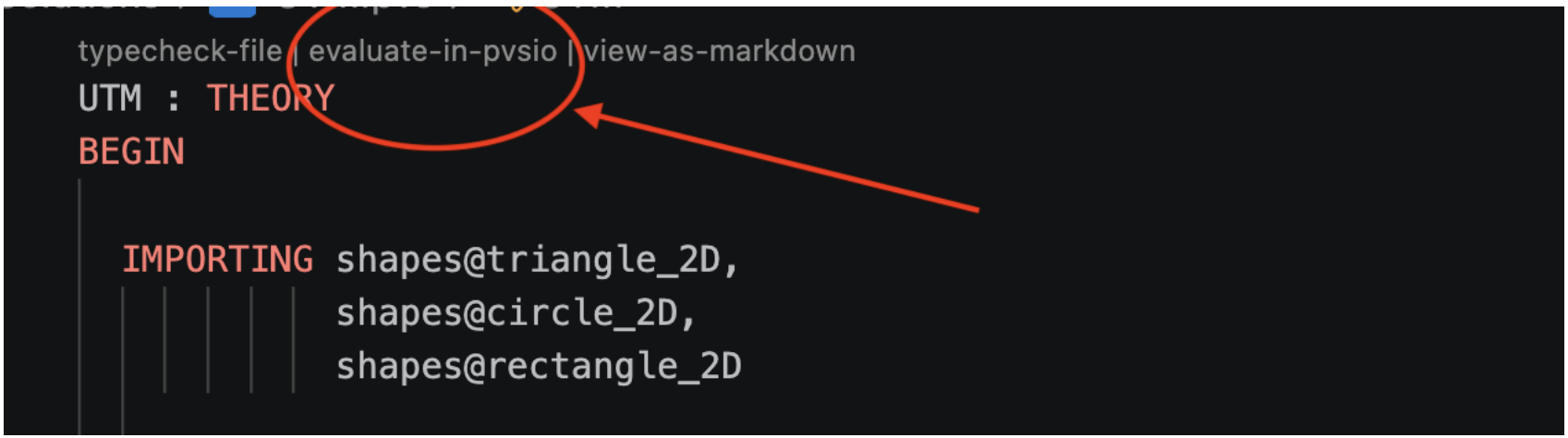
<PVSio> in_operational_volume(USSC)(0,0,-100);
==>
FALSE

<PVSio> in_operational_volume(USSC)(-2000,2000,0);
==>
FALSE
```

(Animation of Functional Specifications . . .

PVSio is

- a *read-eval-loop* interface to the PVS Ground Evaluator;
- an efficient and sound mechanism to compute within the theorem prover;
- available as the standalone Unix command `pvsio`, in Emacs, through `M-x pvsio`, and in VSCode-PVS.



Input/Output Operations

```
<PVSio> 1/7;
==>
0.142857

<PVSio> sqrt(2);
==>
1.414213...

<PVSio> LET r = abs(read_real) IN
print("sqrt("+r+")="+sqrt(r));
10
sqrt(10)=3.162277

<PVSio> printf("Hello World ~s",today);
Hello World "05/10/2026"
```

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==>
1.414213...

<PVSio> LET r = abs(read_real) IN
print("sqrt("+r+")="+sqrt(r));
10
sqrt(10)=3.162277

<PVSio> printf("Hello World ~s",today);
Hello World "05/10/2026"
```

Random Values

```
<PVSio> RANDOM;
==>
0.277062...

<PVSio> RANDOM=RANDOM;
==>
FALSE

<PVSio> LET r=RANDOM IN r=r;
==>
TRUE
```

Random Values

```
<PVSio> RANDOM;
==>
0.277062...

<PVSio> RANDOM=RANDOM;
==>
FALSE

<PVSio> LET r=RANDOM IN r=r;
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TRUE
```

Random Values

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<PVSio> RANDOM;
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0.277062...

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==>
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<PVSio> LET r=RANDOM IN r=r;
==>
TRUE
```

PVSio Extensions Are Sound

- From the point of view of the type-checker and theorem prover, PVSio extensions are uninterpreted functions.
- PVSio extensions are not evaluated by the theorem prover,
- For PVS, `RANDOM` is just an uninterpreted constant and, therefore, `RANDOM = RANDOM` always hold.

More PVSio Features

- I/O files.
- Mutable and global variables.
- Exceptions.
- Type introspection.
- Unbounded loops (possibly non-terminating).
- Extensionality via user-defined semantic attachments.

PVSio Utilities

- PVSioChecker: Differential testing.
- PVSioCSV: Reader and writer of comma-separated value files.
- PVSioRegex: Regular expressions matcher.
- PVSioPEG: Lexer/parser via Parsing Expression Grammars (PEG).
- PVSioJSON: Reader and writer of JSON files.
- PVSioKeyVal: Persistent key/value store of PVS values.
- PVS2C: Memory efficient C generator for PVS specifications functional specifications.

. . . References)

- Website: `http://shemesh.larc.nasa.gov/fm/pvs/PVSio`.
- *Rapid prototyping in PVS*, C. Muñoz, NASA Contract Report.
- *Efficiently Executing PVS*, N. Shankar, SRI Technical Report.
- *Evaluating, Testing, and Animating PVS Specifications*, J. Crow, S. Owre, J. Rushby, N. Shankar, and D. Stringer-Calvert, SRI Technical Report.